On Ordinal Ranks of Baire Class Functions

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Abstract Harmonic Analysis
June 2018
National Sun Yat-sen University

Outline

1. Review
   - Baire Class Functions
   - Ordinal Ranks
2. Main Results
   - $\beta_2^* \approx \gamma_2^*$
   - β_2^* is not essentially multiplicative
3. Open Problem

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   - β_2^* is not essentially multiplicative
3. Open Problem

Baire Class Functions

X Polish space (separable and completely metrizable).

Definition.

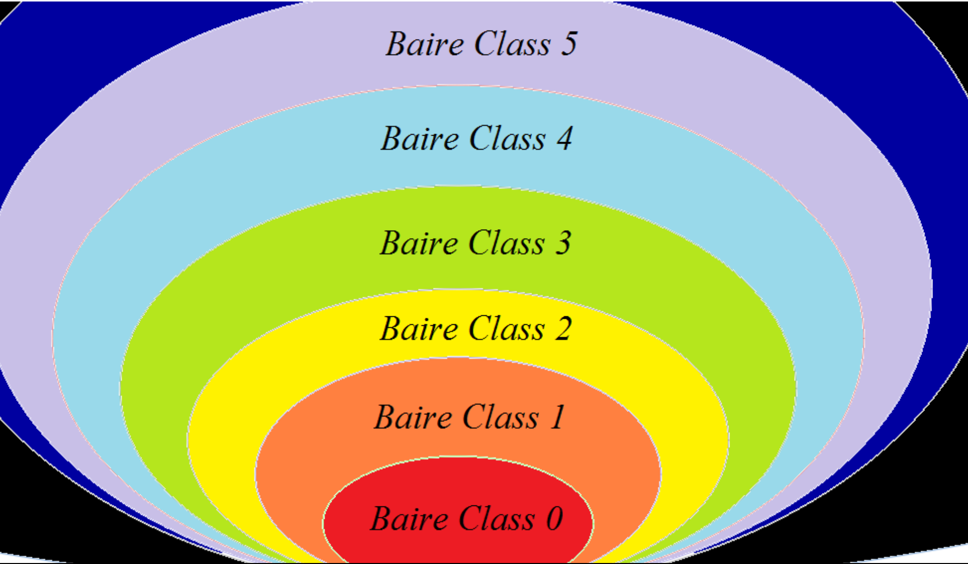
$f : X \to \mathbb{R}$ is said to be **Baire Class** 1 *if it is a pointwise limit of a sequence of continuous functions.*

$\xi \geq 1$ countable ordinal

Definition.

$f : X \to \mathbb{R}$ is said to be **Baire Class** ξ *if it is a pointwise limit of a sequence of Baire Class ζ functions where $\zeta < \xi$*

Baire Hierarchy



Ordinal Ranks

Settings:

1. X Polish space
2. $\xi \geq 1$ countable ordinal
3. $\mathfrak{B}_\xi = \{f : X \to \mathbb{R} : f \text{ Baire Class } \xi\}$

Definition.

$\rho : \mathfrak{B}_\xi \to \omega_1$ *ordinal rank*

Purpose: measure complexity

Equivalent Definitions of Baire Class 1 functions

X Polish

Theorem.

$f : X \to \mathbb{R}$ *is* **Baire Class** 1 *if it satisfies one of the following.*

1. $f_{|F}$ *has a point of continuity for every non-empty closed subset* $F \subseteq X$ *(oscillation rank)*
2. f *is a pointwise limit of continuous functions (convergence rank)*
3. $f^{-1}(U)$ *is* F_σ *for every open* $U \subseteq X$ *(separation rank)*

(1) $\Leftrightarrow$ (2): Baire
(2) $\Leftrightarrow$ (3): Lebesgue, Hausdorff and Banach

Oscillation Rank β

Recall: $\beta : \mathfrak{B}_1 \to \omega_1$ *ordinal rank*

$f : X \to \mathbb{R}$ is a Baire Class 1 function and $\varepsilon > 0$. Recall that

$$osc_\alpha(f, x) = \inf_{\substack{x \in U \\ U \text{ open}}} \sup_{y,z \in U \cap D^\alpha} |f(y) - f(z)|.$$

$$D^0 = X$$

$$D^1 = \{x \in X : osc_0(f, x) \geq \varepsilon\}$$ (ε-discontinuities)

$$D^2 = \{x \in D^1 : osc_1(f, x) \geq \varepsilon\}$$ (limit points of ε- discontinuities)

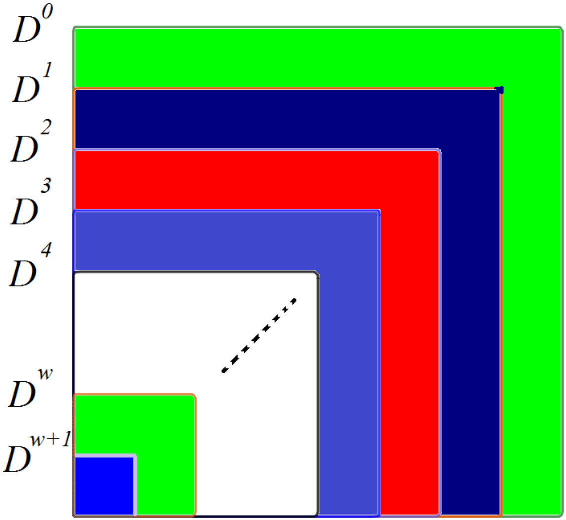
$$D^3 = \{x \in D^2 : osc_2(f, x) \geq \varepsilon\}$$ (limit points of D^2)

$$\vdots$$

$$D^\omega = \cap_{n=1}^{\infty} D^n$$

$$\beta(f, \varepsilon) = \min\{\eta : D^\eta = \emptyset\} \qquad \beta(f) = \sup\{\beta(f, \varepsilon) : \varepsilon > 0\}$$

Picture of D^α sets



Examples of β

Recall that

$$\beta(f,\varepsilon) = \min\{\eta : D^\eta = \emptyset\} \qquad \beta(f) = \sup\{\beta(f,\varepsilon) : \varepsilon > 0\}$$

Examples

1. f is continuous $\Rightarrow \beta(f) = 1$
2. $X = \mathbb{R}$

$$A = \{0\} \qquad \Rightarrow \beta(\chi_A) = 2$$

$$A = \{0\} \cup \bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\} \qquad \Rightarrow \beta(\chi_A) = 3$$

$$A = \{0\} \cup \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left(\left\{\frac{1}{n}\right\} \cup \left\{\frac{1}{n} + \frac{1}{m}\right\}\right) \qquad \Rightarrow \beta(\chi_A) = 4$$

3. f is Baire Class 1 $\Rightarrow \beta(f) < \omega_1$

Convergence Rank γ

Assume that $(f_n)_{n\in\mathbb{N}}$ sequence of functions on X and $\varepsilon > 0$

$$NUC_\alpha((f_n),x) = \inf_{\substack{x\in U \\ U \text{ open}}} \inf_{N\in\mathbb{N}} \sup\{|f_m(y)-f_n(y)| : y \in U\cap D^\alpha, n, m \geq N\}$$

$$D^0 = X$$

$$D^1 = \{x \in X : NUC_0((f_n),x) \geq \varepsilon\}$$ (non ε-uniformly convergent)

$$D^2 = \{x \in D^1 : NUC_1((f_n),x) \geq \varepsilon\}$$

$$D^3 = \{x \in D^2 : NUC_2((f_n),x) \geq \varepsilon\}$$

$$\vdots$$

$$D^\omega = \cap_{n=1}^\infty D^n$$

$$\gamma((f_n),\varepsilon) = \min\{\eta : D^\eta = \emptyset\}$$

$$\gamma(f) = \min\{\sup_{\varepsilon>0} \gamma((f_n),\varepsilon) : f_n \text{ continuous}, f_n \to f \text{ pointwise}\}$$

Examples of γ

$$NUC_\alpha((f_n),x) = \inf_{\substack{x\in U\\ U \text{ open}}} \inf_{N\in\mathbb{N}} \sup\{|f_m(y) - f_n(y)| : y \in U \cap D^\alpha, n, m \geq N\}$$

1. f is continuous $\Rightarrow \gamma(f) = 1$
2. $A = \{0\} \Rightarrow \gamma(\chi_A) \geq 2$ $(\beta(f) \leq \gamma(f))$
3. f is Baire Class 1 $\Rightarrow \gamma(f) < \omega_1$

Generalized β_2^* and γ_2^* Ranks

Recall: $\mathcal{F}_\sigma(\tau)$ is a collection of all F_σ sets with respect to τ
(X, τ) is a Polish space, $f : X \to \mathbb{R}$ Baire Class 2 function
Define

$$T_{f,2} = \{\tau' : \tau' \text{ is Polish }, \tau \subseteq \tau' \subseteq \mathcal{F}_\sigma(\tau), f \in \mathfrak{B}_1(\tau')\}$$

Definition (Elekes, Kiss, Vidnyánszky 16').

$$\beta_2^*(f) = \min\{\beta_{\tau'}(f) : \tau' \in T_{f,2}\}$$
$$\gamma_2^*(f) = \min\{\gamma_{\tau'}(f) : \tau' \in T_{f,2}\}$$

where $\beta_{\tau'}(f)$ and $\gamma_{\tau'}(f)$ are $\beta(f)$ and $\gamma(f)$ in τ' topology.

Márton Elekes, Viktor Kiss and Zoltán Vidnyánszky, Ranks on the Baire class ξ functions, Trans. Amer. Math. Soc. **368**(2016), 8111-8143.

Kuratowski's Theorem

$$T_{f,2} = \{\tau' : \tau' \text{ is Polish }, \tau \subseteq \tau' \subseteq \mathcal{F}_\sigma(\tau), f \in \mathfrak{B}_1(\tau')\} \neq \emptyset ?$$

Theorem (Kuratowski).

Assume that

1. (X, τ) *Polish space*
2. $A \in \mathcal{F}_\sigma(\tau) \cap \mathcal{G}_\delta(\tau)$

Then there is a Polish topology τ' *such that*

1. $\tau \subseteq \tau' \subseteq \mathcal{F}_\sigma(\tau)$
2. A *is clopen in* τ'

Recall that

1. $\mathcal{F}_\sigma(\tau)$ is a collection of all F_σ sets with respect to τ
2. $\mathcal{G}_\delta(\tau)$ is a collection of all G_δ sets with respect to τ

Solved Problems

For any $f, g \in \mathfrak{B}_2$,

1. $\beta_2^* \approx \gamma_2^*$ (essentially equivalent) if

$$\beta_2^*(f) \lesssim \gamma_2^*(f) \quad \text{and} \quad \gamma_2^*(f) \lesssim \beta_2^*(f)$$

2. β_2^* is essentially multiplicative if

$$\beta_2^*(fg) \leq \max\{\beta_2^*(f), \beta_2^*(g)\}$$

Problem (Elekes, Kiss, Vidnyánszky 16').

1. *$\beta_2^* \approx \gamma_2^*$ on $\mathfrak{B}_2$?*
2. *Are the ranks β_2^* and γ_2^* essentially multiplicative?*

Márton Elekes, Viktor Kiss and Zoltán Vidnyánszky, Ranks on the Baire class ξ functions, Trans. Amer. Math. Soc. **368**(2016), 8111-8143.

Outline

Main results

Theorem (Leung, N., Tang 18').

$\beta_2^* \approx \gamma_2^*$ *on* $\mathfrak{B}_2$

Theorem (Leung, N., Tang 18').

Both β_2^* *and* γ_2^* *are not essentially multiplicative.*

Motivation: $\beta_2^* \approx \gamma_2^*$ on $\mathfrak{B}_2$

Recall that $\beta_2^* \approx \gamma_2^*$ if

$$\beta_2^*(f) \lesssim \gamma_2^*(f) \quad \text{and} \quad \gamma_2^*(f) \lesssim \beta_2^*(f)$$

Corollary (Elekes, Kiss, Vidnyánszky 16').

$\beta_2^* \leq \gamma_2^*$ *on* $\mathfrak{B}_2$

Proposition (Elekes, Kiss, Vidnyánszky 16').

If $(f_n)_{n\in\mathbb{N}}$ *is a sequence of Baire Class* 1 *functions converging uniformly to* f, *then* $\gamma(f) \lesssim \sup_n \gamma(f_n)$.

Key Proposition: $\beta_2^* \approx \gamma_2^*$ on $\mathfrak{B}_2$

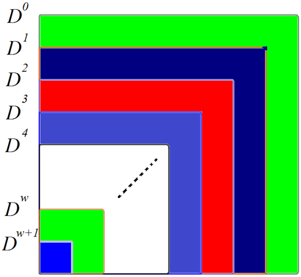
Proposition (Leung, Tang 03').

Assume that

1. X *Polish space*
2. $F \subseteq X$ *closed subset*
3. $f : X \to \mathbb{R}$ *Baire Class* 1
4. $\varepsilon > 0$

Then there exists $g : F \setminus D^1(f, \varepsilon, F) \to \mathbb{R}$ *such that*

1. g *is continuous*
2. $|f(x) - g(x)| < \varepsilon$ *for all* $x \in F \setminus D^1(f, \varepsilon, F)$



Trick: Partition of unity

Key Proposition: $\beta_2^* \approx \gamma_2^*$ on $\mathfrak{B}_2$

Proposition.

Assume that

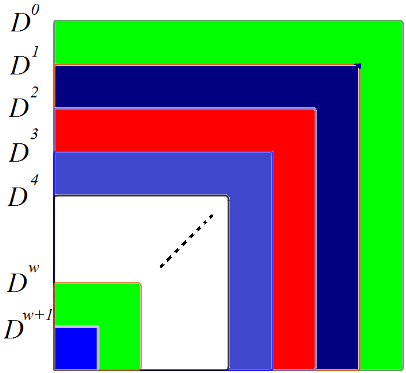
1. $f : X \to \mathbb{R}$ *Baire Class* 1
2. $\varepsilon > 0$

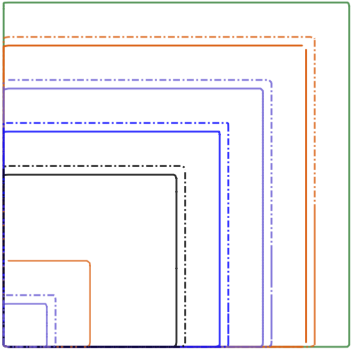
Then there exists $g : X \to \mathbb{R}$ *such that*

1. g *is Baire Class* 1
2. $\gamma(g) \leq \beta(f)$
3. $|f(x) - g(x)| < \varepsilon$ *for all* $x \in X$

Trick: For each 'piece', perform 'gluing' to obtain g.

Main Result: $\beta_2^* \approx \gamma_2^*$ on $\mathfrak{B}_2$





g Baire Class 1, $|f-g|<\varepsilon$ on X and $\gamma(g)\leq\beta(f)$

Theorem (Leung, N., Tang 18').

$\beta_2^* \approx \gamma_2^*$ on $\mathfrak{B}_2$

Motivation: β_2^* is not essentially multiplicative

Recall: β_2^* is essentially multiplicative if for any $f, g \in \mathfrak{B}_2$,

$$\beta_2^*(fg) \leq \max\{\beta_2^*(f), \beta_2^*(g)\}$$

Theorem (Elekes, Kiss, Vidnyánszky 16').

β_2^* is unbounded on the set of characteristics functions.

Recall: $\beta_2^* : \mathfrak{B}_2 \to \omega_1$

Key Propositions: β_2^* is not essentially multiplicative

Proposition.

Let (X, τ) be an uncountable Polish space. Then there exists a subset $A \subseteq X$ such that $\chi_A \in \mathfrak{B}_2(\tau)$ and $\omega < \beta_2^*(\chi_A) \leq \omega + 2$.

Key Propositions: β_2^* is not essentially multiplicative

Proposition.

Assume that

1. (X, τ) *Polish space*
2. τ' *is a Polish topology on* X *such that* $\tau \subseteq \tau' \subseteq \mathcal{F}_\sigma(\tau)$
3. $\chi_A \in \mathfrak{B}_1(\tau')$ *such that* $\beta_{\tau'}(\chi_A) \leq \omega \cdot 2$

Then there exist

1. *Polish topology* τ'' *on* X *such that* $\tau' \subseteq \tau'' \subseteq \mathcal{F}_\sigma(\tau)$
2. *function* $g : X \to \mathbb{N}$ *such that* $\beta_{\tau''}\left(\frac{\chi_A}{g}\right) \leq \omega$ *and* $\beta_{\tau''}(g) \leq 2$.

Main trick: $\mathcal{F}_\sigma(\tau)$ has generalized reduction property.

Main result

Theorem (Leung, N., Tang 18').

β_2^* is not essentially multiplicative. Particularly, if (X, τ) is an uncountable Polish space, then there exist

1. *$f \in \mathfrak{B}_2(\tau)$ such that $\beta_2^*(f) \leq \omega$ and*
2. *$g \in \mathfrak{B}_2(\tau)$ such that $\beta_2^*(g) \leq \omega$*

but $\beta_2^*(fg) > \omega$.

Theorem (Leung, N., Tang 18').

γ_2^* is not essentially multiplicative.

Outline

Open Problem

(X, τ) uncountable Polish space

Proposition.

Assume that

1. $\zeta \geq 1$ *ordinal*

Then

1. ζ *countable ordinal* $\Rightarrow$ *there exists* $\chi_A \in \mathfrak{B}_\xi(\tau)$ *such that* $\zeta < \beta^*_\xi(\chi_A) \leq \zeta + 2$.
2. ζ *limit ordinal* $\Rightarrow$ *there exists* $f \in \mathfrak{B}_\xi(\tau)$ *such that* $\beta^*_\xi(f) = \zeta$.

Open Problem

Problem.

Assume that

1. (X, τ) *be an uncountable Polish space*
2. $\xi \geq 2$ *be a countable ordinal.*

Is it true that for any nonzero countable ordinal ζ, *there exists* $f \in \mathfrak{B}_\xi(\tau)$ *such that* $\beta^*_\xi(f) = \zeta$*?*

References

- D. H. Leung, H,-W, Ng and W.-K. Tang, On Ordinal Ranks of Baire Class Functions, preprint, arXiv: 1701.05649v1
- Márton Elekes, Viktor Kiss and Zoltán Vidnyánszky, Ranks on the Baire class ξ functions, Trans. Amer. Math. Soc. **368**(2016), 8111-8143.
- D. H. Leung and W.-K. Tang, Functions of Baire class one, Fund. Math. **179**(2003), 225-247.

Thank you for your time.