Applications of multi-norms to group algebras

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Banach modules

Let A be a Banach algebra. Then a **Banach left A-module** is a left A-module E such that $(E, \|\cdot\|)$ is a Banach space and

$$\|a \cdot x\| \leq \|a\| \, \|x\| \quad (a \in A, \, x \in E).$$

This is denoted by: $E \in A\text{–}\mathbf{mod}$. Similarly for $E \in \mathbf{mod}\text{–}A$ and $E \in A\text{–}\mathbf{mod}\text{–}A$

Thus $E \in A\text{–}\mathbf{mod}$ iff there is a continuous homomorphism $\rho : A \to \mathcal{B}(E)$.

Examples

1) E is a closed left ideal in A; A itself is an A-bimodule.

2) E is a Banach algebra containing A as a closed subalgebra.

3) $A \widehat{\otimes} E$ for a Banach space E. $\square$

Morphisms

Let E and F be Banach spaces. Then an operator $T \in \mathcal{B}(E,F)$ is **admissible** if $\ker T$ and $T(E)$ are complemented in E and F, respectively.

Let E and F be left A-modules. Then a **morphism** is a linear map such that

$$T(a \cdot x) = a \cdot Tx \quad (a \in A,\, x \in E).$$

Let A be a Banach algebra, and $E, F \in A\text{-}\mathbf{mod}$. Then ${}_A\mathcal{B}(E,F)$ is the closed linear subspace of $\mathcal{B}(E,F)$ consisting of the left A-module morphisms

Example There exists $\pi \in \mathcal{B}(A \widehat{\otimes} E, E)$ with

$$\pi(a \otimes x) = a \cdot x \quad (a \in A,\, x \in E)\,.$$

Then $\pi \in {}_A\mathcal{B}(A \widehat{\otimes} E, E)$. $\square$

Operators as modules

Let E be any Banach space, and let A be a Banach algebra. For $a, b \in A$, and $T \in \mathcal{B}(A, E)$, define

$$(a \cdot T)(b) = T(ba), \quad (T \cdot a)(b) = T(ab)$$

Then $\mathcal{B}(A, E) \in A\text{–}\mathbf{mod}\text{–}A$.

Define $\Pi : E \to \mathcal{B}(A, E)$ by

$$\Pi(x)(a) = a \cdot x \quad (a \in A, x \in E)\,.$$

Then $\Pi \in {}_A\mathcal{B}(E, \mathcal{B}(A, E))$.

Connection: Let $E \in A-\mathbf{mod}$, so that the dual $E' \in \mathbf{mod}-A$. Then the dual module of $A \widehat{\otimes} E$ is $\mathcal{B}(A, E')$ with the prescribed module operations, and the dual of $\pi \in \mathcal{B}(A \widehat{\otimes} E, E)$ is

$$\pi' = \Pi \in \mathcal{B}(E', \mathcal{B}(A, E'))\,.$$

Projectivity

Let $P \in A\text{--}\mathbf{mod}$. Then P is **projective** if, for each $E, F \in A\text{--}\mathbf{mod}$, for each admissible epimorphism $T \in {}_A\mathcal{B}(E, F)$, and for each $S \in {}_A\mathcal{B}(P, F)$, there exists $R \in {}_A\mathcal{B}(P, E)$ with $T \circ R = S$. Thus R **lifts** S.

Example: Set $P = A \widehat{\otimes} E$. Then P is a **free** Banach left A-module. Easy: P is projective in $A\text{--}\mathbf{mod}$. $\square$

Test: Let $E \in A\text{--}\mathbf{mod}$. Then E is projective if and only if there exists $\rho \in {}_A\mathcal{B}(E, A \widehat{\otimes} E)$ with $\pi \circ \rho = I_E$ (so that π is a **retraction**).

Eg: A unital. Then take $\rho(a) = a \otimes e_A$. So A is projective in $A\text{--}\mathbf{mod}$.

Injectivity

Let $J \in A$–$\mathbf{mod}$. Then J is **injective** if, for each $E, F \in A$–$\mathbf{mod}$, for each admissible monomorphism $T \in {}_A\mathcal{B}(E, F)$, and for each $S \in {}_A\mathcal{B}(E, J)$, there exists $R \in {}_A\mathcal{B}(F, J)$ with $R \circ T = S$.

Example: Let E be a Banach space. Then $\mathcal{B}(A, E)$ is a **cofree** Banach left A-module. Easy: $\mathcal{B}(A, E)$ is injective in A–$\mathbf{mod}$. $\square$

Test: Let $E \in A$–$\mathbf{mod}$, and suppose that $\{x \in E : A \cdot x = \{0\}\} = \{0\}$. Then E is injective if and only if there exists $\rho \in {}_A\mathcal{B}(\mathcal{B}(A, E), E)$ with $\rho \circ \Pi = I_E$ (so that Π is a **coretraction**).

Flat modules

Suppose that E is projective - so that there exists $\rho \in {}_A\mathcal{B}(E, A \widehat{\otimes} E)$ with $\pi \circ \rho = I_E$. Then

$$\rho' \in \mathcal{B}_A((A \widehat{\otimes} E)', E') = \mathcal{B}_A(\mathcal{B}(A, E'), E')$$

with $\rho' \circ \Pi = I_{E'}$.

It follows that the dual E' of a projective left A-module E is an injective right A-module.

Let $E \in A\text{–}\mathbf{mod}$. Then E is **flat** if E' is injective in $\mathbf{mod}\text{–}A$.

(The original definition was different; we say **biflat** in the category $A\text{–}\mathbf{mod}\text{–}A$.)

Basic Theorem [B. E. Johnson] Let A be an amenable Banach algebra, and $E \in A\text{–}\mathbf{mod}$ or $E \in \mathbf{mod}\text{–}A$. Then E' is injective, equivalently E is flat. $\square$

Group algebras

Let G be a locally compact group, with left Haar measure m, and let $L^1(G) = L^1(G, m)$ be the group algebra of G (with convolution product). The dual space of $L^1(G)$ is $L^\infty(G)$, the Banach space of essentially bounded functions on G. This space contains the constant function 1. For $\lambda \in L^\infty(G)$ and $s \in G$, define a translate $s \cdot \lambda \in L^\infty(G)$ by

$$\langle f, s \cdot \lambda \rangle = \langle L_s f, \lambda \rangle,$$

where $(L_s f)(t) = f(st)$ $(s, t \in G)$.

An element $\Lambda \in L^\infty(G)' = L^1(G)''$ is a **left translation-invariant mean** if $\|\Lambda\| = \langle \Lambda, 1 \rangle = 1$, and $\langle \Lambda, s \cdot \lambda \rangle = \langle \Lambda, \lambda \rangle \quad (s \in G, \lambda \in L^\infty(G))$.

Definition The group G is **amenable** if there is a left translation-invariant mean on $L^\infty(G)$.

Amenability of group algebras

There is always one character on the algebra $L^1(G)$; this is the **augmentation character** φ_G, defined by

$$\varphi_G : f \mapsto \int_G f(t)\,\mathrm{d}m(t), \quad L^1(G) \to \mathbb{C}\,.$$

Theorem [B. E. Johnson] Let G be a locally compact group. Then the following are equivalent:

(a) the Banach algebra $L^1(G)$ is amenable;

(b) the locally compact group G is amenable;

(c) the module $\mathbb{C}_{\varphi_G}$ is flat in $L^1(G)$–**mod**;

(d) the closed ideal $\ker \varphi_G$ has a bounded approximate identity. □

Modules over $L^1(G)$

Set $A = L^1(G)$. We can take the following in the category A–**mod**:

- $E = A$;

- $E = L^p(G)$ for $1 < p < \infty$ and convolution product, so that E is a dual module [eg, $p = 2$];

- $E = A' = L^\infty(G)$, with dual module operation given by

$$(f \cdot \lambda)(t) = \int_G f(s)\lambda(ts)\,\mathrm{d}m(s);$$

- $E = M(G)$, the measure algebra on G, with product

$$(\mu \star \nu)(B) = \int_G \nu(s^{-1}B)\,\mathrm{d}\mu(s)$$

for each Borel subset B of G, so that A is a closed ideal in $M(G)$, and $M(G) = C_0(G)'$ in A–**mod**–A.

When are they projective/injective/flat?

Partial answers (mainly [DP1])

Set $A = L^1(G)$.

Case 1 $E = A$ is always projective.

Note that $A \,\widehat{\otimes}\, A = L^1(G \times G)$. Take a compact K in G with $m(K) = 1$, and define a map $\rho \in {}_A\mathcal{B}(A, A \,\widehat{\otimes}\, A)$ by

$$\rho(f)(s,t) = \chi_K(t^{-1})f(st)$$

for $f \in A$ and $s, t \in G$. Then $\pi \circ \rho = I_A$.

Case 2 $E = M(G)$. Then $M(G)$ is projective iff G is discrete. We can also prove that $M(G)$ is flat whenever G is discrete or amenable.

I **guess** that $M(G)$ is always flat in A–$\mathbf{mod}$.

More partial answers

Case 3 $E = L^\infty(G) = A'$

Theorem Suppose that $L^\infty(G)$ is projective in A–$\mathbf{mod}$. Then G is finite.

Proof Use the fact that π is a retraction to show that $C_0(G)$ is complemented in $L^\infty(G)$. By a theorem of Lau–Losert, the space $C_0(G)$ is complemented in $L^\infty(G)$ only when G is finite. □

Case 4 $E = L^\infty(G)' = A''$.

Theorem Suppose that $L^\infty(G)'$ is projective in A–$\mathbf{mod}$. Then G is discrete and contains no infinite, amenable subgroup. □

Guess: In fact, G must be finite.

Case 5 Take p with $1 < p < \infty$. Then $L^p(G)$ is projective in A–$\mathbf{mod}$ if and only if G is compact.

Amenability and injectivity

We aim for a converse to the statement:

'A amenable implies each $E \in A$–$\mathbf{mod}$ is flat'.

Definition
Let E be a Banach left $L^1(G)$-module. An element $\lambda \in E'$ is an **augmentation-invariant** functional if

$$\langle f \cdot x, \lambda \rangle = \varphi_G(f) \langle x, \lambda \rangle \quad (f \in L^1(G),\, x \in E)\,.$$

The module E is **augmentation-invariant** if there is a non-zero, augmentation-invariant functional on E.

Examples (i) $E = L^\infty(G)$ is augmentation-invariant if and only if G is amenable.

(ii) $M(G)$ is always augmentation-invariant. (Take $\lambda = \varphi_G : \mu \mapsto \mu(G)$.)

(iii) $L^\infty(G)'$ is always augmentation-invariant. (Take λ to be the constant function $1 \in L^\infty(G)$.) $\square$

Some injectivity results

Theorem Let E be the dual of a Banach right $L^1(G)$-module. Suppose that E is faithful and augmentation-invariant. Then E is injective if and only if G is amenable.

Proof Suppose that $E = F'$ is injective. Start with an augmentation-invariant functional $\lambda_0 \in E'$ and $x_0 \in E$ with $\langle x_0, \lambda \rangle = 1$, set $T_0 = \Pi(x_0)$, and note that $\rho(T_0) = x_0$. Use weak toplogies, dualities, Mazur to find a net (h_α) in $P(G)$ that satisfies Reiter's condition for amenability. $\square$

Corollary Let G be a locally compact group. (1) The following are equivalent:

(a) $M(G)$ is injective;

(b) $L^\infty(G)$ is flat;

(c) G is amenable.

(2) $L^1(G)$ is injective iff G is discrete and amenable. $\square$

(p, q)-multi-norms

Now G is a locally compact group, and we take p, q with $1 \leq p \leq q < \infty$.

Definition Let G be a locally compact group, and take p, q with $1 \leq p \leq q < \infty$. A functional $\Lambda \in L^\infty(G)'$ is **left** (p, q)**-multi-invariant** if the set $\{s \cdot \Lambda : s \in G\}$ is multi-bounded with respect to the (p, q)-multi-norm.

The group G is **left** (p, q)**-amenable** if there exists a left (p, q)-multi-invariant mean in $L^\infty(G)'$.

Using results about dominance of (p, q)-multi-norms on $L^1(\Omega, \mu)$, we see that, for a mean $\Lambda \in L^\infty(G)'$, we have

left-invariant $\Rightarrow$ left (q, q)-invariant

$\Rightarrow$ left (p, q)-invariant

$\Leftrightarrow$ left $(1, q)$-invariant.

We need a converse.

A first theorem

Theorem Let G be a locally compact group, and take p, q with $1 \leq p \leq q < \infty$. Then G is amenable if and only if G is left (p, q)-amenable.

Proof Take Λ to be a left (p, q)-multi-invariant mean on $L^\infty(G)$. Then $\{s \cdot \Lambda : s \in G\}$ is (p, q)-multi-bounded. By an earlier theorem, it is weakly compact. So its closed convex hull, say K, is weakly compact. For each $s \in G$, the map $L_s : \Phi \mapsto s \cdot \Phi$ is an isometric affine map on K, and these maps form a group. By Ryll–Nardzewski, the family has common fixed point. This is an invariant mean on $L^\infty(G)$. So G is amenable. $\square$

Another $L^1(G)$-module

Let G be a locally compact group, and take p with $1 < p < \infty$. Define

$$J = \mathcal{B}(L^1(G), L^p(G))\,.$$

Define an action of G on J by

$$(t * U)\,(f) = t \cdot U(t^{-1} \cdot f) \quad (f \in L^1(G), U \in J).$$

The map $t \mapsto (t * U)(f)$ is continuous, and $J \in L^1(G)$–$\mathbf{mod}$ for the operation

$$(g * U)\,(f) = \int_G g(t)\,(t * U)\,(f)\,\mathrm{d}m(t)$$

for $f, g \in L^1(G)$ and $U \in J$.

An embedding

Define an embedding $\Pi : L^p(G) \to J$ by

$$(\Pi(g)(f) = \varphi_G(f)g \quad (f \in L^1(G), g \in L^p(G)) .$$

This is a left $L^1(G)$-module morphism, and it is admissible. (A left inverse is $U \mapsto U(f_0)$ for any $f_0 \in L^1(G)$ with $\varphi_G(f_0) = 1$.)

Theorem Let G be a locally compact group, and take p with $1 < p < \infty$. Suppose that $L^p(G)$ is injective in $L^1(G)-\mathbf{mod}$. Then the morphism Π is a coretraction: that is, there is $R \in_{L^1(G)} \mathcal{B}(J, L^p(G))$ with $R \circ \Pi$ the identity on $L^p(G)$.

Proof Easy from the definition of injectivity. □

Two technical lemmas

Let Ω be a measure space. To give a flavour of the calculations; recall that $\|\cdot\|_n^{[q]}$ is the standard q-multi-norm on $L^q(\Omega)$.

Lemma 1 Let E be a Banach space and take p, q with $1 \leq p \leq q < \infty$. Then, for each $\Phi_1, \ldots, \Phi_n \in E''$, we have

$$\|(\Phi_1, \ldots, \Phi_n)\|_n^{(p,q)} = \sup \left\|(T''(\Phi_1), \ldots, T''(\Phi_n))\right\|_n^{[q]},$$

where the supremum is taken over all operators $T \in \mathcal{B}(E, L^p(\Omega))_{[1]}$. $\square$

Lemma 2 Take $U \in B(L^1(\Omega), L^p(\Omega))$, and set $q = p'$. Suppose that $f_1, \ldots, f_n \in L^q(\Omega)$ have disjoint supports and $g_1, \ldots, g_n \in L^p(\Omega)$ have disjoint supports. Set

$$T = \sum_{i=1}^{n} U'(f_i) \otimes g_i, \quad L^1(\Omega) \to L^p(\Omega)\,.$$

Then

$$\|T\| \leq \|U\| \max\{\|f_i\|_q \|g_i\|_p : i = 1, \ldots, n\}\,. \quad \square$$

The main theorem

Theorem Let G be a locally compact group, and take p with $1 < p < \infty$. Then $L^p(G)$ is injective in $L^1(G)-\mathbf{mod}$ if and only if G is amenable.

Proof Take J and R as above. For each compact V in G with $m(V) > 0$, define Λ_V by

$$\langle \lambda, \Lambda_V \rangle = \frac{1}{m(V)} \int_V (R(\lambda \otimes \chi_V))(t) \, \mathrm{d}m(t)$$

for $\lambda \in L^\infty(G)$. Then $\Lambda_V \in L^\infty(G)'$ and $\|\Lambda_V\| \leq \|R\|$.

We can suppose that (Λ_V) converges weak-$*$ in $L^\infty(G)'$, say to Λ. Since $\langle 1, \Lambda \rangle = 1$, Λ is non-zero.

We *claim* that (a multiple of) Λ is left (p, p)-multi-invariant. If so, G is amenable by an earlier theorem.

Proof continued

Take distinct $s_1, \ldots, s_n \in G$. Choose V so that $s_1 V, \ldots, s_n V$ are pairwise-disjoint. Take $U \in J$, and let $\{X_1, \ldots, X_n\}$ be a measurable partition of G. Take $f_1, \ldots, f_n \in L^q(G)_{[1]}$, where $q = p'$, such that supp$f_i \subset X_i$, and set

$$T = \sum_{i=1}^{n} U'(f_i) \otimes \chi_{s_i V}, \quad L^1(G) \to L^p(G)\,.$$

By Lemma 2, $\|T\| \leq \|U\|\, m(V)^{1/p}$.

More calculations show that

$$\left(\sum_{i=1}^{n} \left\| \chi_{X_i} U''(s_i \cdot \Lambda) \right\|_n^{(p,q)} \right)^{1/p} \leq \|R\|\ .$$

By Lemma 1,

$$\|(s_1 \cdot \Lambda, \ldots, s_n \cdot \Lambda)\|_n^{(p,p)} \leq \|R\|\ .$$

This gives the claim. □

Summary

Theorem Let G be a locally compact group, and take p with $1 < p < \infty$. Then the following are equivalent:

(a) G is amenable;

(b) $L^1(G)$ is an amenable Banach algebra;

(c) $L^p(G)$ is injective;

(d) $L^p(G)$ is flat;

(e) G is left (p, q)-amenable for all $q \geq p$;

(f) G is left (p, q)-amenable for some $q \geq p$;

(g) G is left $(1, q)$-amenable for all $q \geq 1$. $\square$

Further comments

1) The specific theorem was also proved by G. Racher by more direct methods.

2) Let S be a left-cancellative semigroup, and take $p \geq 1$. Then $\ell^p(S)$ is injective in $\ell^1(S)$–$\mathbf{mod}$ iff S is an amenable group.

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