Introduction to multi-norms

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Congratulations to Antony To-Ming Lau on
his upcoming 75^{th} birthday
and on the
2018 David Borwein Distinguished Career
Award

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References

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Basic definitions

Let $(E, \|\cdot\|)$ be a normed space. The closed unit ball is $E_{[1]}$ and the dual space is E'.

A **multi-norm** on $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is a norm on E^n, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following hold for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in E$:

(A1) $\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\right\|_n = \|(x_1, \ldots, x_n)\|_n$ for each permutation σ of $\{1, \ldots, n\}$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n$

$$\leq (\max_{i \in \mathbb{N}_n} |\alpha_i|) \|(x_1, \ldots, x_n)\|_n$$

for each $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$;

(A3) $\|(x_1, \ldots, x_n, 0)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$;

(A4) $\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$.

See [**DP2**].

Dual multi-norms

For a **dual multi-norm**, replace (A4) by:

(B4) $\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_{n-1}, 2x_n)\|_n$.

Let $(\|\cdot\|_n)$ be a multi-norm or dual multi-norm based on a space E. Then we have a **multi-normed space** and a **dual multi-normed space**, respectively. They are **multi-Banach spaces** and **dual multi-Banach spaces** when E is complete.

Let $\|\cdot\|_n$ be a norm on E^n. Then $\|\cdot\|_n'$ is the dual norm on $(E^n)'$, identified with $(E')^n$.

The **dual** of $(E^n, \|\cdot\|_n)$ is $((E')^n, \|\cdot\|_n')$. The dual of a multi-normed space is a dual multi-Banach space; the dual of a dual multi-normed space is a multi-Banach space. See [**DP2**].

What are multi-norms good for?

1) Solving some specific questions - for example, characterizing when some modules over group algebras are injective [**DDPR1 and second talk**].

2) Understanding the geometry of Banach spaces that goes beyond the shape of the unit ball.

3) Throwing light on absolutely summing operators

4) Giving a theory [**DP2, DLOT**] of 'multi-bounded linear operators' between Banach spaces.

5) Giving results about Banach lattices [**DP2, DLOT**].

6) Giving a theory of decompositions [**DP2**] of Banach spaces generalizing known theories.

7) Possible generalizations to 'Multi-Banach algebras'; $L^1(G)$ is a good example.

Minimum and maximum multi-norms

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space or a dual multi-normed space. Then

$$\max \|x_i\| \le \|(x_1, \ldots, x_n)\|_n \le \sum_{i=1}^{n} \|x_i\| \qquad (*)$$

for all $x_1, \ldots, x_n \in E$ and $n \in \mathbb{N}$.

Example 1 Set $\|(x_1, \ldots, x_n)\|_n^{\min} = \max \|x_i\|$. This gives the **minimum** multi-norm.

Example 2 It follows from (*) that there is also a **maximum** multi-norm, which we call $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$.

Note that it is **not** true that $\sum_{i=1}^{n} \|x_i\|$ gives the maximum multi-norm — because it is not a multi-norm. (It is a dual multi-norm.)

A characterization of multi-norms

Give $\mathbb{M}_{m,n}$ a norm by identifying it with $\mathcal{B}(\ell_n^\infty, \ell_m^\infty)$.

Let E be a normed space. Then $\mathbb{M}_{m,n}$ acts from E^n to E^m in the obvious way.

Consider a sequence $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is a norm on E^n and such that $\|x\|_1 = \|x\|$ for each $x \in E$.

Theorem [DP2] This sequence of norms is a multi-norm if and only if

$$\|a \cdot x\|_m \le \|a : \ell_n^\infty \to \ell_m^\infty\| \, \|x\|_n$$

and a dual multi-norm if and only if

$$\|a \cdot x\|_m \le \left\|a : \ell_n^1 \to \ell_m^1\right\| \, \|x\|_n$$

for all $m, n \in \mathbb{N}$, $a \in \mathbb{M}_{m,n}$, and $x \in E^n$. $\square$

p-multi-norms

We could calculate $\|a\|$ in different ways - for example, by identifying $\mathbb{M}_{m,n}$ with $\mathcal{B}(\ell_n^p, \ell_m^p)$ for other values of p. Thus a sequence of norms $(\|\cdot\|_n)$ is a p-**multi-norm** if

$$\|a \cdot x\|_m \leq \|a : \ell_n^p \to \ell_m^p\| \, \|x\|_n$$

for all $m, n \in \mathbb{N}$, $a \in \mathbb{M}_{m,n}$, and $x \in E^n$.

The sequence $(\|\cdot\|_n)$ is a **strong** p-**multi-norm** if $\|\mathbf{y}\|_n \leq \|\mathbf{x}\|_m$ whenever $m, n \in \mathbb{N}$, $\mathbf{x} \in E^m$, $\mathbf{y} \in E^n$, and $\|\langle \mathbf{y}, \lambda \rangle\|_{\ell_n^p} \leq \|\langle \mathbf{x}, \lambda \rangle\|_{\ell_m^p}$ for all $\lambda \in E'$.

A strong p-multi-norm is a p-multi-norm; they are the same iff $p = 2$ or $p = \infty$.

See the memoir **[DLOT]**, which generalizes much of the earlier material.

Another characterization

This is taken from [**DDPR1**]. It gives a 'coordinate-free' characterization.

Let $(E, \|\cdot\|)$ be a normed space. Then a c_0-**norm** on $c_0 \otimes E$ is a norm $\|\cdot\|$ such that:

1) $\|a \otimes x\| \leq \|a\| \, \|x\| \quad (a \in c_0, x \in E)$;

2) $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$ with $\|T \otimes I_E\| = \|T\|$ whenever T is a compact operator on c_0;

3) $\|\delta_1 \otimes x\| = \|x\| \quad (x \in E)$.

Each c_0-norm is a reasonable cross-norm; we can replace 'T is compact' by 'T is bounded'.

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The connection

Theorem Multi-norms on $\{E^n : n \in \mathbb{N}\}$ correspond to c_0-norms on $c_0 \otimes E$. The injective tensor product norm gives the minimum multi-norm, and the projective tensor product norm gives the maximum multi-norm $\square$

The recipe is: given a c_0-norm $\|\cdot\|$, set

$$\|(x_1, \dots, x_n)\|_n = \left\| \sum_{j=1}^{n} \delta_j \otimes x_j \right\| \quad (x_1, \dots, x_n \in E)\,.$$

Thus the theory of multi-norms could be a theory of norms on tensor products.

There is a similar identification of p-multi-norms with $\ell^p \otimes E$ in **[DLOT]**.

Helemski has a generalization to $L^p(\Omega) \otimes E$.

Banach lattice multi-norms

Let $(E, \|\cdot\|)$ be a complex Banach lattice.

Examples $L^p(\Omega)$, $L^\infty(\Omega)$, or $C(K)$ with the usual norms and the obvious lattice operations are all (complex) Banach lattices.

Definition [**DP2**] Let $(E, \|\cdot\|)$ be a Banach lattice. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, set

$$\|(x_1, \ldots, x_n)\|_n^L = \| \, |x_1| \vee \cdots \vee |x_n| \, \|$$

and

$$\|(x_1, \ldots, x_n)\|_n^{DL} = \| \, |x_1| + \cdots + |x_n| \, \| \, .$$

Then $(E^n, \|\cdot\|_n^L)$ is a multi-Banach space. It is the **Banach lattice multi-norm**. Also $(E^n, \|\cdot\|_n^{DL})$ is a dual multi-Banach space. It is the **dual Banach lattice multi-norm**.

Each is the dual of the other.

The canonical lattice p-multi-norm

Let E be a Banach lattice, and take p with $1 \leq p \leq \infty$. Set

$$\|(x_1, \ldots, x_n)\|_n^{L,p} = \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\|$$

for $x_1, \ldots, x_n \in E$. Then the sequence $(\| \cdot \|_n^{L,p})$ is a strong p-multi-norm.

We recover lattice multi-norms with $p = \infty$ and dual lattice multi-norms with $p = 1$.

A representation theorem

Clause (1) below is basically a theorem of **Pisier**, as given in a thesis of a student, **Marcolino Nhani**. Our approach is different. The main theorem of **[DLOT]** is a similar representation theorem for strong p-multi-norms; the word 'strong' is essential.

Theorem

(1) Let $(E^n, \|\cdot\|_n)$ be a multi-Banach space. Then there is a Banach lattice X such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $(Y^n, \|\cdot\|_n^L)$ for a closed subspace Y of X.

(2) Let $(E^n, \|\cdot\|_n)$ be a dual multi-Banach space. Then there is a Banach lattice X such that $(E^n, \|\cdot\|_n)$ is multi-isometric to $((X/Y)^n, \|\cdot\|_n^{DL})$ for a closed subspace Y of X. $\square$

Summing norms - I

Take $p \in [1, \infty)$. For $x_1, \dots, x_n \in E$, set

$$\mu_{p,n}(x_1, \dots, x_n) = \sup_{\lambda \in E'_{[1]}} \left\{ \left(\sum_{j=1}^{n} \left| \langle x_j, \lambda \rangle \right|^p \right)^{1/p} \right\}.$$

This is the **weak p-summing norm**. For example, we can see that

$$\mu_{1,n}(x_1, \dots, x_n) = \sup \left\{ \left\| \sum_{j=1}^{n} \zeta_j x_j \right\| : \zeta_1, \dots, \zeta_n \in \mathbb{T} \right\}.$$

For $\lambda_1, \dots, \lambda_n \in E'$, we have

$$\mu_{1,n}(\lambda_1, \dots, \lambda_n) = \sup \left\{ \sum_{j=1}^{n} \left| \langle x, \lambda_j \rangle \right| : x \in E_{[1]} \right\}.$$

Theorem [DP2] The dual of $\| \cdot \|_n^{\max}$ is $\mu_{1,n}$. $\square$

The (p,q)–multi-norm

Let E be a Banach space, and take p,q with $1 \leq p \leq q < \infty$. Define

$$\|(x_1,\dots,x_n)\|_n^{(p,q)} = \sup\left\{\left(\sum_{j=1}^n \left|\langle x_j,\lambda_j\rangle\right|^q\right)^{1/q}\right\},$$

taking the sup over all $\lambda_1,\dots,\lambda_n \in E'$ with $\mu_{p,n}(\lambda_1,\dots,\lambda_n) \leq 1$.

Fact [DP2]: $\{(E^n, \|\cdot\|_n^{(p,q)}) : n \in \mathbb{N}\}$ is a multi-Banach space.

Then $(\|\cdot\|_n^{(p,q)})$ is the (p,q)**–multi-norm** based on E.

Remarks (1) The $(1,1)$-multi-norm is the maximum multi-norm based on E.

(2) The (p,q)–multi-norm over E'', when restricted to E, is the (p,q)–multi-norm over E.

The standard t-multi-norm on $L^r(\Omega)$

Let Ω be a measure space, and take r, t with $1 \leq r \leq t < \infty$. We consider the Banach space $L^r(\Omega)$, with the usual L^r-norm $\| \cdot \|$. (Think of ℓ^r.)

For each family $\mathbf{X} = \{X_1, \dots, X_n\}$ of pairwise-disjoint measurable subsets of Ω such that $X_1 \cup \cdots \cup X_n = \Omega$, we set

$$r_{\mathbf{X}}((f_1, \dots, f_n)) = \left(\left\| P_{X_1} f_1 \right\|^t + \cdots + \left\| P_{X_n} f_n \right\|^t \right)^{1/t} ,$$

where $P_X : L^r(\Omega) \to L^r(X)$ is the natural projection.

Finally, $\|(f_1, \dots, f_n)\|_n^{[t]} = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \dots, f_n))$.

This is the **standard t-multi-norm** (on $L^r(\Omega)$) from [**DP2**].

Remark Let $t = r$.Then

$$\|(f_1, \dots, f_n)\|_n^{[r]} = \| \, |f_1| \vee \cdots \vee |f_n| \, \| \, ,$$

which is the lattice multi-norm on $L^r(\Omega)$.

Summing norms - II

Again $1 \leq p \leq q < \infty$, and E and F are Banach spaces. For $T \in \mathcal{B}(E,F)$, $\pi_{q,p}^{(n)}(T)$ is

$$\sup\left\{\left(\sum_{j=1}^{n}\left\|Tx_j\right\|^q\right)^{1/q} : \mu_{p,n}(x_1,\ldots,x_n) \leq 1\right\}.$$

Definition Let $T \in \mathcal{B}(E,F)$. Suppose that

$$\pi_{q,p}(T) := \lim_{n\to\infty} \pi_{q,p}^{(n)}(T) < \infty\,.$$

Then T is (q,p)-**summing**; the set of these is $\Pi_{q,p}(E,F)$. This gives a Banach space.

The much-studied space $\Pi_{q,p}(E,F)$ is a component of an **operator ideal**: see the book of Pietsch.

A connection

Let E be a normed space. Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, and define

$$T_{\boldsymbol{x}} : (\zeta_1, \ldots, \zeta_n) \mapsto \sum_{j=1}^{n} \zeta_j x_j \,, \quad \mathbb{C}^n \to E \,.$$

Then $\mu_{p,n}(\boldsymbol{x}) = \left\| T_{\boldsymbol{x}} : \ell_n^{p'} \to E \right\|$ for $p \geq 1$.

It follows that

$$\|\boldsymbol{x}\|_n^{(p,q)} = \pi_{q,p}(T'_{\boldsymbol{x}} : E' \to c_0) \,.$$

This leads to:

Theorem Let E be a normed space, and suppose that $1 \leq p \leq q < \infty$. Then the (p,q)-multi-norm induces the norm on $c_0 \otimes E$ given by embedding $c_0 \otimes E$ into $\Pi_{q,p}(E', c_0)$. □

Multi-bounded sets and operators

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. A subset B of E is **multi-bounded** if

$$c_B := \sup_{n\in\mathbb{N}} \{\|(x_1, \ldots, x_n)\|_n : x_1, \ldots, x_n \in B\} < \infty .$$

Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be multi-Banach spaces. An operator $T \in \mathcal{B}(E, F)$ is **multi-bounded** if $T(B)$ is multi-bounded in F whenever B is multi-bounded in E. The set of these is a linear subspace $\mathcal{M}(E, F)$ of $\mathcal{B}(E, F)$; $\mathcal{M}(E)$ is a Banach algebra.

Theorem An operator $T \in \mathcal{B}(E, F)$ is multi-bounded iff it is 'multi-continuous'. $\square$

For $T_1, \ldots, T_n \in \mathcal{M}(E, F)$, set

$$\|(T_1, \ldots, T_n)\|_{mb,n} = \sup\{c_{T_1(B)\cup\cdots\cup T_n(B)} : c_B \le 1\} .$$

Theorem Now $(\mathcal{M}(E, F)^n, \|\cdot\|_{mb,n})$ is a multi-Banach space, and $(\mathcal{M}(E)^n, \|\cdot\|_{mb,n})$ is a 'multi-Banach algebra'. $\square$

Regular operators

An operator between Banach lattices is **regular** if it is a linear combination of positive operators.

Theorem Let $E = \ell^p$ and $F = \ell^q$, where $p, q \geq 1$. Regard them as multi-normed spaces with the standard (p, p)- and (q, q)-multi-norms, respectively. Then $\mathcal{M}(E, F)$ consists exactly of the regular operators. $\square$

There are several other theorems in **[DLOT]** relating multi-bounded operators to known classes of operators between Banach lattices.

(p, q)-multi-bounded sets

Suppose that $1 \leq p \leq q < \infty$, and take a Banach space E.

Definition The space $\mathcal{B}_{p,q}(\ell^1, E)$ is the subspace of $\mathcal{B}(\ell^1, E)$ consisting of the operators T such that $\{T(\delta_k) : k \in \mathbb{N}\}$ is (p, q)-multi-bounded in E.

It is a Banach space with respect to the multi-bounded norm.

Theorem Take $T \in \mathcal{B}(\ell^1, E)$. Then $T \in \mathcal{B}_{p,q}(\ell^1, E)$ iff $T' \in \Pi_{q,p}(E', \ell^\infty)$.

Proof Calculations using the definitions. $\square$

Theorem Each (p, q)-multi-bounded set in E is relatively weakly compact.

Proof Use the above theorem. By the Pietsch factorization theorem, every p-summing operator is weakly compact, so T', and hence T, are weakly compact. Then use Eberlein-Šmulian. $\square$

Equivalences of multi-norms

Definition [DP2] Let $(E, \|\cdot\|)$ be a normed space. Suppose that both $(\|\cdot\|_n^1)$ and $(\|\cdot\|_n^2)$ are multi-norms on E. Then $(\|\cdot\|_n^2)$ **dominates** $(\|\cdot\|_n^1)$, written $(\|\cdot\|_n^1) \preccurlyeq (\|\cdot\|_n^2)$, if there is a constant $C > 0$ such that

$$\|\boldsymbol{x}\|_n^1 \leq C \, \|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^{\,n}, \, n \in \mathbb{N}) \, .$$

The two multi-norms are **equivalent**, written

$$(\|\cdot\|_n^1) \cong (\|\cdot\|_n^2)$$

if each dominates the other.

Interpretation in terms of summing operators

Theorem (**DDPR2**) Let E be a normed space. Then $(\|\cdot\|_n^{(p_1,q_1)}) \cong (\|\cdot\|_n^{(p_2,q_2)})$ if and only if $\Pi_{q_1,p_1}(E', c_0) = \Pi_{q_2,p_2}(E', c_0)$ as subsets of $\mathcal{B}(E', c_0)$. $\square$

Thus the theory of equivalence of these multi-norms could be a theory of (q, p)-summing operators.

We wish to decide when various pairs of multi-norms are mutually equivalent - for example, what about (p, q)-multi-norms on ℓ^r?

For detailed calculations which give an almost complete solution, see **[BDP]**.

A triangle with curves

Look at the ‘triangle’

$$\mathcal{T} = \{(p,q) : 1 \le p \le q < \infty\}.$$

For $c \in [0,1)$, look at the curve $\mathcal{C}_c$:

$$\mathcal{C}_c = \left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} = c\right\}.$$

Take $r \in (1,\infty)$. Then the curve $\mathcal{C}_{1/r}$ meets the line $p = 1$ at the point $(1, r')$. The union of these curves is $\mathcal{T}$.

Sample solution: The case where $r \geq 2$

Theorem [BDP] Take $r \geq 2$ and $E = \ell^r$. Then the triangle $\mathcal{T}$ decomposes into the following (mutually disjoint) equivalence classes:

- $\mathcal{T}_{\text{min}} := A_r = \{(p,q) \in \mathcal{T}\colon 1/p - 1/q \geq 1/2\}$;

- the curves $\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c\colon 1 \leq p \leq 2\}$, for $c \in [0, 1/2)$;

- the singletons $\mathcal{T}_{(p,q)} := \{(p,q)\}$ for $(p,q) \in \mathcal{T}$ with $p > 2$.

The solutions are not quite complete in the case where $1 < r < 2$.

Key inequalities: generalized Hölder, Khinchine, Grothendieck; Rademacher functions.

Multi-Banach algebras

Let $(A, \|\cdot\|)$ be a Banach algebra, and let

$$((A^n, \|\cdot\|_n) : n \in \mathbb{N})$$

be a multi-normed space. Then $(A^n, \|\cdot\|_n)$ is a **multi-Banach algebra** if multiplication is a multi-bounded bilinear operator, and so

$$\|(a_1 b_1, \ldots, a_n b_n)\|_n \leq \|(a_1, \ldots, a_n)\|_n \, \|(b_1, \ldots, b_n)\|_n \, .$$

Examples (1) Each Banach algebra is a multi-Banach algebra with respect to both the minimum and maximum multi-norms.

(2) Take $1 \leq p \leq q < \infty$. Then $(\ell^p, \cdot\,)$ is a multi-Banach algebra with respect to the standard (p, q)-multi-norm.

(3) Let G be a locally compact group. Then the group algebra $(L^1(G), \star)$ with the standard $(1,1)$-multi-norm is a multi-Banach algebra.

(4) For each multi-Banach space $(E^n, \|\cdot\|_n)$, $(\mathcal{M}(E)^n, \|\cdot\|_{\mathrm{mb},n})$ is a multi-Banach algebra. $\square$