Local convexity in the space of measurable functions

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General Question: What does local convexity on a subset do?

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- A subset S of $L^0(\mathbb{P})$ is *bounded in probability* if it is a bounded subset of the TVS $L^0(\mathbb{P})$. Same as

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- If (f_n) is a sequence in $L^0(\mathbb{P})$, and $g_k \in \mathrm{co}(f_n)_{n=k}^{\infty}$ for all k, then (g_n) is a sequence of forward convex combinations (FCCs) of (f_n).

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1. All FCCs of (f_n) converges to f in probability.
2. The $L^0(\mathbb{P})$-topology is locally convex on $\mathrm{co}((f_n) \cup \{f\})$.
3. There exists $\mathbb{Q} \sim \mathbb{P}$ such that (f_n) is $L^1(\mathbb{Q})$-bounded and that $\|f_n - f\|_{L^1(\mathbb{Q})} \to 0$.

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1. The $L^0(\mathbb{P})$-topology is locally convex on K.
2. There exists $\mathbb{Q} \sim \mathbb{P}$ such that K is bounded in $L^1(\mathbb{Q})$ and that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on K.
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Example. Let $K = \{f \in L^1_+(\mathbb{P}) : \int f \, d\mathbb{P} = 1\}$. Then K satisfies (2) but not (3).

Nonpositive sets & "de-switching"

[Branath-Schachermayer. LNM 1999] Let K be a convex set in $L^0_+(\mathbb{P})$ that is bounded in probability. Then there exists $\mathbb{Q} \sim \mathbb{P}$ so that K is a bounded set in $L^1(\mathbb{Q})$.

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We generalize the questions above to *bounded* convex sets in $L^1(\mathbb{P})$.

It is also convenient to eliminate the switching of probabilities.

Proposition. Let K be a convex bounded set in $L^1(\mathbb{P})$. Consider the following conditions.

1. There exists $\mathbb{Q} \sim \mathbb{P}$ such that K is bounded in $L^1(\mathbb{Q})$ and that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on K.
2. For any $\varepsilon > 0$, there is a measurable set A with $\mathbb{P}(A) > 1 - \varepsilon$ so that $\|(f_n - f)\chi_A\|_{L^1(\mathbb{P})} \to 0$ for any $f_n, f \in K$ so that $f_n \to f$ in probability.
3. There exists $\mathbb{Q} \sim \mathbb{P}$ such that K is $\mathbb{Q}$-uniform integrable.
4. For any $\varepsilon > 0$, there is a measurable set A with $\mathbb{P}(A) > 1 - \varepsilon$ so that $K_A = \{f\chi_A : f \in K\}$ is $\mathbb{P}$-uniformly integrable.

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Remark. To get (2), it suffices to obtain the following:
For any measurable A with $\mathbb{P}(A) > 0$, there exists measurable $B \subseteq A$ with $\mathbb{P}(B) > 0$ so that $\|(f_n - f)\chi_B\|_{L^1(\mathbb{P})} \to 0$ for any $f_n, f \in K$ so that $f_n \to f$ in probability.

An application of Komlos' Theorem

Theorem. [Komlos. Acta MAS Hung. 1967] Let (f_n) be a bounded sequence in $L^1(\mathbb{P})$, there is a subsequence (f_{n_k}) and a function $f \in L^1(\mathbb{P})$ so that $(\frac{1}{m}\sum_{k=1}^{m} f_{n_k})_m$ converges a.e. to f.

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In particular, "Yes" for Q2.

Locally convex solid

Aim: To characterize the condition that there exists $\mathbb{Q} \sim \mathbb{P}$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on K, where K is convex bounded in $L^1(\mathbb{P})$.

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Definition. Let S be a nonempty subset of K. We say that the $L^0(\mathbb{P})$-topology is *uniformly locally convex solid* on S if for each $L^0(\mathbb{P})$-neighborhood U of 0, there is a convex solid set $W \subseteq U$ such that for each $f \in S$, $(f + W) \cap K$ is a neighborhood of f for the restriction of the $L^0(\mathbb{P})$-topology to K.

A separation theorem and its consequence

Theorem. Let K be a convex bounded set in $L^1(\mathbb{P})$ and let S be a nonempty subset of K. Assume that the $L^0(\mathbb{P})$-topology is uniformly locally convex solid on S. If A is a measurable set with $\mathbb{P}(A) > 0$, then there exists $0 \neq g \in L^\infty_+(\mathbb{P})$, $\operatorname{supp} g \subseteq A$ such that

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Idea: Find a sequence of convex solid sets W_k and $r > 0$ so that

1. For each $f \in S$, $(f + W_k) \cap K$ is a a neighborhood of f for the restriction of the $L^0(\mathbb{P})$-topology to K.
2. g is a linear functional that separates $rB_{L^1(\mathbb{P})}$ and kW_k on one side and χ_A on the other.

A local Hahn-Banach Theorem

The same ideas can be used to prove the following:

Theorem. Let (X, τ) be a real Hausdorff TVS. Let K be a convex circled set in X. Suppose that the restriction of τ to K is locally convex (at 0). The set of all linear functionals on X that are τ-continuous on K separates points of K.

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Remark. If K is also circled, then the $L^0(\mathbb{P})$-topology is uniformly locally convex solid on K if and only if it is locally convex solid at 0.

Corollary. Let K be convex solid and bounded in $L^1(\mathbb{P})$. TFAE.

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Choose a solid neighborhood of 0 in $L^0(\mathbb{P})$, V, so that $V \cap K \subseteq C \cap K$.
Since $V \cap K$ is solid, $W = \text{co}(V \cap K)$ is a solid convex set contained in $C \subseteq U$ and $W \cap K$ is a neigborhood of 0 in K.

A counterexample

Example. (Based on [Pryce, P Edin MS, 1972]) There is a bounded convex circled set K in $L^1(0,1)$ that is $L^0(\mathbb{P})$-compact so that the $L^0(\mathbb{P})$-topology is locally convex on K, but there does not exists $\mathbb{Q} \sim \mathbb{P}$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on K.

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Set $Y_n = F_n(X_n)$ and

$$K = \{\sum a_n Y_n : \sum |a_n| \leq 1\}.$$

Back to the positive case

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Corollary. Let K be a bounded convex set in $L^1_+(\mathbb{P})$. Assume that the $L^0(\mathbb{P})$-topology is locally convex on K. Then for any $f \in K$ and any $\varepsilon > 0$, there is a measurable set A with $\mathbb{P}(A) > 1 - \varepsilon$ so that $\|(f_n - f)\chi_A\|_{L^1(\mathbb{P})} \to 0$ for any sequence (f_n) in K that converges to f in probability.

Corollary. Let K be a bounded convex set in $L^1_+(\mathbb{P})$. Assume that the $L^0(\mathbb{P})$-topology is locally convex on K. Let S be a countable set in K. Then for any $\varepsilon > 0$, there is a measurable set A with $\mathbb{P}(A) > 1 - \varepsilon$ so that $\|(f_n - f)\chi_A\|_{L^1(\mathbb{P})} \to 0$ for any sequence (f_n) in K that converges to some $f \in S$ in probability.

Corollary. Let K be a bounded convex set in $L^1_+(\mathbb{P})$. Assume that the $L^0(\mathbb{P})$-topology is locally convex on K. Let S be a countable set in K. Then for any $\varepsilon > 0$, there is a measurable set A with $\mathbb{P}(A) > 1 - \varepsilon$ so that $\|(f_n - f)\chi_A\|_{L^1(\mathbb{P})} \to 0$ for any sequence (f_n) in K that converges to some $f \in S$ in probability.

The following is a sort of generalization of [Kadaras-Zitkovic].

Proposition. Let (f_n) be a bounded sequence in $L^1_+(\mathbb{P})$ and let $K = \mathrm{co}(f_n)$. If the $L^0(\mathbb{P})$-topology is locally convex on K, then there exists $\mathbb{Q} \sim \mathbb{P}$ such that the $L^0(\mathbb{Q})$- and $L^1(\mathbb{Q})$-topologies agree on K.

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Thus $\|(g_k - g)\chi_A\|_{L^1(\mathbb{P})} \to 0$.

Thank You