

[雙週一題]網路數學問題徵答  
九十五學年度第二學期

主辦單位：中山大學應用數學系  
補助單位：教育部

第八題：

96.06.15公佈，96.06.29中午12點截止

化簡下式成一有理數

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2n + mn^2 + 2mn}$$

解答：

$$\begin{aligned}
 \text{原式} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn(m+n+2)} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{1}{m(m+n+2)} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \left[ \frac{1/(n+2)}{m} - \frac{1/(n+2)}{m+n+2} \right] \\
 &= \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \left[ \left(1 - \frac{1}{n+3}\right) + \left(\frac{1}{2} - \frac{1}{n+4}\right) + \left(\frac{1}{3} - \frac{1}{n+5}\right) + \cdots \right] \\
 &= \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \left[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+2} \right] \\
 &= \sum_{n=1}^{\infty} \left( \frac{1/2}{n} - \frac{1/2}{n+2} \right) \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n+2} \right) \\
 &= \frac{1}{2} \left[ \left(1 - \frac{1}{3}\right) \left(1 + \frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \right. \\
 &\quad \left. + \left(\frac{1}{3} - \frac{1}{5}\right) \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) + \cdots \right] \\
 &= \frac{1}{2} \left[ \left(1 + \frac{1}{2} + \frac{1}{3}\right) + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \frac{1}{3} \left(\frac{1}{4} + \frac{1}{5}\right) \right. \\
 &\quad \left. + \frac{1}{4} \left(\frac{1}{5} + \frac{1}{6}\right) + \frac{1}{5} \left(\frac{1}{6} + \frac{1}{7}\right) + \cdots \right] \\
 &= \frac{1}{2} \left[ \frac{11}{6} + \frac{1}{2} \cdot \frac{25}{12} + \left( \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6} + \cdots \right) + \left( \frac{1}{3 \times 5} + \frac{1}{4 \times 6} + \frac{1}{5 \times 7} + \cdots \right) \right] \\
 &= \frac{1}{2} \left[ \frac{11}{6} + \frac{25}{24} + \left( \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots \right) \right. \\
 &\quad \left. + \frac{1}{2} \left( \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots \right) \right] \\
 &= \frac{1}{2} \left[ \frac{11}{6} + \frac{25}{24} + \frac{1}{3} + \frac{1}{6} + \frac{1}{8} \right] \\
 &= \frac{1}{2} \left[ \frac{44 + 25 + 8 + 7}{24} \right] = \frac{84}{48} = \frac{7}{4}
 \end{aligned}$$

答案請寄至－高雄市中山大學應數系圖書館的『雙週一題』信箱，或傳真07-5253809，或利用電子郵件信箱problem@math.nsysu.edu.tw（主旨為「雙週一題」）。解答上請註明姓名、校名、校址縣市、系所、年級、班級、學號和E-mail。