[雙週一題]網路數學問題徵答
九十三學年度第一學期

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大專組：

令$x_1, x_2, ..., x_{10}$爲實數，且$1 \leq x_i \leq 2$。
試求$(x_1 + x_2 + ... + x_{10})(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_{10}})$的最大值與最小值。

【解答】

Let $S = (x_1 + x_2 + ... + x_{10})(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_{10}})$.
(i)By Cauchy-Schwarz inequality, we have $S \geq (\underbrace{1 + 1 + ... + 1}_{\text{10 terms}})^2 = 100$

(ii)We have $S = 10 + \sum_{i<j}(\frac{x_i}{x_j} + \frac{x_j}{x_i})$.
Let $a = \frac{x_i}{x_j}$, then $\frac{1}{2} \leq a \leq 2$.
And let $f(a) = a + \frac{1}{a}$,
then $f(a)$ is increasing on $[1, 2]$ and is decreasing on $[\frac{1}{2}, 1]$.
that is, $\frac{x_i}{x_j} + \frac{x_j}{x_i} = a + \frac{1}{a} \leq 2 + \frac{1}{2}$,
and equality is obtained iff one of the pair x_i, x_j equals 1, the other 2.
Thus, S is maximal iff one half of the x_i equal 1, the other half equal 2,
thus $S = (\frac{10}{2} \cdot 1 + \frac{10}{2} \cdot 2)(\frac{10}{2} \cdot 1 + \frac{10}{2} \cdot \frac{1}{2}) = \frac{225}{2}$.

【註】成績於11月30日前公布於雙週一題網頁。(www.math.nsysu.edu.tw/∼problem)