Approximate and exact D-optimal designs for multiresponse polynomial regression models

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Abstract

The D-optimal design problems in polynomial regression models with a one-dimensional control variable and k-dimensional response variable $Y = (Y_1, \cdots, Y_k)$ where there are some common unknown parameters are discussed. The approximate D-optimal designs are shown to be independent of the covariance structure between the k responses when the degrees of the k responses are of the same order. Then, the exact n-point D-optimal designs are also discussed. Krafft and Schaefer (1992) and Imhof (2000) are useful in obtaining our results. We extend the proof of symmetric cases for $k \geq 2$.

Keywords and phrases: parallel line assay, D-optimal design, multiple response, exact design.

1 Introduction

In Finney (1978), statistical methods in biological assay are introduced and illustrated. To be more explicit, a biological assay is an experiment for estimating the potency of a material, by means of the reaction that follows its application to living matter. A typical comparative experiment in biology involves applying known treatments to subjects, measuring the subjects, and then estimating the differences between the effects of the treatments. The aim of a biological assay is to use the measurements as a foundation for comparing the potencies of the treatments. This new aim affects both the optimal experimental design and statistical analysis.

Biological assay seeks to estimate equally effective doses of the standard and test preparations, that is to say doses whose inverse ratio will estimate the potency of the test preparation relative to the standard. Suppose that a subject receives a dose z of a particular stimulus, and that the response subsequently measured is u. The average or expected response to the dose may be written

$$F(z) = E(u|z)$$

Let S denote indicator of the standard preparation and T denote indicator of the test preparation. If two preparations contain the same effective constituent in fixed proportions and all other constituents are without effect on $F(z)$. The two regression functions must be related by

$$F_T(z) = F_S(\rho z)$$

where for all z, ρ is constant, the potency of T relative to S. A response, Y, may be assumed to have linear effect

$$E(Y_S|z) = \alpha + \beta x,$$

where $x = \log(z)$. Similarity ensures that the same metameters linearized both the S and T regression functions. Moreover, T may have

$$\begin{cases} E(Y_T|z) = \alpha + \beta\rho^\lambda x, & \lambda \neq 0; \\ E(Y_T|z) = \alpha + \beta\log\rho + \beta x, & \lambda = 0. \end{cases}$$

where the most common values of λ are 0 and 1. If $\lambda = 0$, the lines are parallel, the assay is then called parallel line assays.

If the dose-response regression can not be put into the form of $\alpha + \beta x$, or of some simple modification involving perhaps an extra parameter as in logistic function, the assay problem is more complicated. When S has a quadratic regression

$$E(Y_S|z) = \alpha + \beta x + \gamma x^2,$$

that for T it may take the troublesome form

$$\begin{cases} E(Y_T|z) & = & \alpha + \beta\rho^{\lambda}x + \gamma\rho^{2\lambda}x^2, & \lambda & \neq & 0; \\ E(Y_T|z) & = & \alpha + \beta(\log\rho + x) + \gamma(\log\rho + x)^2, & \lambda & = & 0. \end{cases}$$

We will discuss the design problems model for parameter estimation in parallel line assay. The general k dose-response regression models of degree m extended from parallel line assays can be expressed as

$$\begin{cases} E(Y_1|z) & = & \theta_{0,1} + \theta_{1,1}x + \cdots + \theta_{m-1,1}x^{m-1} + \theta_m x^m \\ E(Y_2|z) & = & \theta_{0,2} + \theta_{1,2}x + \cdots + \theta_{m-1,2}x^{m-1} + \theta_m x^m \\ & \vdots & \\ E(Y_k|z) & = & \theta_{0,k} + \theta_{1,k}x + \cdots + \theta_{m-1,k}x^{m-1} + \theta_m x^m \end{cases}$$

where $x = \log(z)$, with $Cov(Y) = \Sigma$.

2 Preliminary

Consider the general multiresponse models of degree m with some common parameters

$$\begin{cases} y_{1j} &= \eta_1(x_j, \vartheta) + \epsilon_{1j} \\ y_{2j} &= \eta_2(x_j, \vartheta) + \epsilon_{2j} \\ &\vdots \\ y_{kj} &= \eta_k(x_j, \vartheta) + \epsilon_{kj}, \ j = 1, \cdots, n \end{cases}$$

where y_{ij} reprsents the jth observation on the ith response function defined on the design space $\Omega = [a, b]$. The response function $\eta_i(x, \vartheta), i = 1, \cdots, k$, are assumed to be known but with unknown parameter vector $\vartheta = (\theta_1, \cdots, \theta_m)'$, and ϵ_{ij} is a random error with $E(\epsilon_{ij}) = 0; E(\epsilon_{ij}^2) = \sigma_i^2; E(\epsilon_{ij}\epsilon_{pj}) = \sigma_{ip}, i \neq p$ and $E(\epsilon_{ij}\epsilon_{pq}) = 0, j \neq q$.

An approximate design ξ is a probability measure on Ω with finite support,

$$\xi = \begin{pmatrix} x_1, & \cdots & , x_r \\ w_1, & \cdots & , w_r \end{pmatrix}$$

with distinct support points $x_1, \cdots, x_r \in \Omega$, and weights $1 > w_1, \cdots, w_r > 0$, $\sum_{i=1}^r w_i = 1$, and $r \in \aleph$.

If all the weights w_i are integral multiples of $1/n$ for a given $n \in \aleph$, then ξ is called an exact design with n observations, and we write $\xi_n = \xi$.

If $\eta_i(x, \vartheta)$, $i = 1, \cdots, k$, are linear regression functions and can be represented as $\eta_i(x, \vartheta) = f_i'(x)\vartheta$, where $f_i(x)$ is the regression function corresponding to $\eta_i(x, \vartheta)$, $i = 1, \cdots, k$. Let $F(x) = [f_1(x), \cdots, f_k(x)]$ be a $m \times k$ matrix, then the information matrix of a design ξ is

$$M(\xi) = \int_\Omega F(x)\Sigma^{-1}F'(x)d\xi(x),$$

where Σ is the common covariance matrix of $\epsilon_j = (\epsilon_{1j}, \cdots, \epsilon_{kj})'$. Let $\hat{\vartheta}$ be the least square estimates of the parameter ϑ, then

$$Cov(\hat{\vartheta}) \propto M^{-1}(\xi)$$

A design ξ^* is called D-optimal if ξ^* maximizes the determinant of the information matrix $M(\xi)$ among all possible design,

$$|M(\xi^*)| = \max_{\xi} |M(\xi)|$$

where it is assumed that on there exists an approximate design with nonsingular information matrix.

Denote E and E_n as the sets of all approximate designs and all exact designs with n observations respectively. An approximate design $\xi^* \in E$ is called D-optimal in E, iff it maximizes $detM(\xi)$ over $\xi \in E$, and an exact design $\xi_n^* \in E_n$ is called D-optimal in E_n, iff it maximizes $detM(\xi_n)$ over $\xi_n \in E_n$.

Krafft and Schaefer (1992) considers a linear regression model with a one-dimensional control variable and multiresponse variables, where the parameters in each response are different. Under rather mild assumptions on the set of regression functions a factorization lemma has been given that the multiresponse D-optimal is independent of the covariance matrix of the response variables. Bischoff (1995) has also discussed some similar problems. Huang and Luh (1999) has discussed the case where there are some common parameters in each response model in the low degree. The problem we study here also has to restriction that it is assumed there are some common parameters in each response model, it is not restricted in the low degree.

The exact D-optimal design problem for polynomial regression on a compact interval has attracted a lot of attention in the literature. Salaevskii (1966) conjectures that an exact D-optimal design ξ^* distributes observations as evenly as possible among the r support points of the approximate D-optimal design. This conjecture is also stated by Wynn (1972). In fact, Hohmannn and Jung (1975) proves the linear case ($m = 1$) and Granovskii (1967) proves the partial solutions for quadratic case ($m = 2$). Gaffke and Krafft (1982) has given Huang (1987) and Gaffke (1987) finds the sufficient conditions for the minimum sample size for the results to hold. Krafft and Schaefer (1992) proves some symmetric case for the individual response variables in a multiresponse polynomial model with first-order or second-order models in each response. Imhof (2000) completes the proofs not solved in

Krafft and Schaefer (1992) for the other cases include symmetric and asymmetric. The problem we study here that exact D-optimal design in a multiresponse polynomial model with the same parameter of quadratic order in each response and extend the results of symmetric cases for $k \geq 2$.

3 Approximate D-optimal designs for k polynomial models

In this section, we discuss the model with the following form. Consider a polynomial regression with k dimensional response variables $Y = (Y_1, Y_2, \cdots, Y_k)'$

$$\begin{cases} E(Y_1) &= \theta_{0,1} + \theta_{1,1}x + \cdots + \theta_{r,1}x^r + \theta_{r+1}x^{r+1} + \cdots + \theta_m x^m \\ E(Y_2) &= \theta_{0,2} + \theta_{1,2}x + \cdots + \theta_{r,2}x^r + \theta_{r+1}x^{r+1} + \cdots + \theta_m x^m \\ &\vdots \\ E(Y_k) &= \theta_{0,k} + \theta_{1,k}x + \cdots + \theta_{r,k}x^r + \theta_{r+1}x^{r+1} + \cdots + \theta_m x^m \end{cases} \tag{3.1}$$

where $x \in \Omega = [-1, 1]$, with $Cov(Y) = \Sigma$.

Let $\vartheta = (\theta_{0,1}, \theta_{1,1}, \cdots, \theta_{r,1}, \theta_{0,2}, \theta_{1,2}, \cdots, \theta_{r,2}, \cdots, \theta_{0,k}, \theta_{1,k}, \cdots, \theta_{r,k}, \theta_{r+1}, \cdots, \theta_m)'$. The model (3.1) can be represented as

$$\begin{cases} \eta_1(x,\vartheta) &= f_1'(x)\vartheta &= (f_{11}'(x), \mathbf{0}, \mathbf{0}, \cdots, \mathbf{0}, f_{22}'(x))\vartheta \\ \eta_2(x,\vartheta) &= f_2'(x)\vartheta &= (\mathbf{0}, f_{11}'(x), \mathbf{0}, \cdots, \mathbf{0}, f_{22}'(x))\vartheta \\ &\vdots \\ \eta_k(x,\vartheta) &= f_k'(x)\vartheta &= (\mathbf{0}, \mathbf{0}, \mathbf{0}, \cdots, f_{11}'(x), f_{22}'(x))\vartheta \end{cases}$$

where $f_{11}(x) = (1, x, \cdots, x^r)'$,$f_{22}(x) = (x^{r+1}, \cdots, x^m)'$, and $\mathbf{0}$ is a $1 \times (m-r)$ vector with all elements 0. These k polynomial regression models have the common parameters $(\theta_{r+1}, \cdots, \theta_m)$, but the coefficient parameters for the lower degree polynomial terms are not the same. Define $F(x) = [f_1(x), \cdots, f_k(x)]$. Thus, the information matrix of a design ξ can be written as

$$\begin{aligned} M(\xi) &= \int_{-1}^{1} F(x)\Sigma^{-1}F'(x)d\xi(x) \\ &= \begin{pmatrix} M_{11}(\xi) & M_{12}(\xi) \\ M_{21}(\xi) & M_{22}(\xi) \end{pmatrix} \end{aligned}$$

where

$$
\begin{aligned}
M_{11}(\xi) &= \int_{-1}^{1} (f_{11}(x) \otimes I_k)\Sigma^{-1}(f'_{11}(x) \otimes I_k) d\xi(x) \\
&= \int_{-1}^{1} (f_{11}(x) f'_{11}(x)) d\xi(x) \otimes \Sigma^{-1} \\
&= A_{11}(\xi) \otimes \Sigma^{-1} \\
M_{12}(\xi) &= \int_{-1}^{1} (f_{11}(x) \otimes I_k)(\mathbf{1} \otimes \Sigma^{-1})(f'_{22}(x) \otimes \mathbf{1}) d\xi(x) \\
&= \int_{-1}^{1} (f_{11}(x) f'_{22}(x)) d\xi(x) \otimes (\Sigma^{-1}\mathbf{1}) \\
&= A_{12}(\xi) \otimes (\Sigma^{-1}\mathbf{1}) \\
M_{21}(\xi) &= A'_{12}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}) \\
M_{22}(\xi) &= \int_{-1}^{1} (f_{22}(x) \otimes \mathbf{1}')(\mathbf{1} \otimes \Sigma^{-1})(f'_{22}(x) \otimes \mathbf{1}) d\xi(x) \\
&= \int_{-1}^{1} (f_{22}(x) f'_{22}(x)) d\xi(x) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1}) \\
&= A_{22}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1})
\end{aligned}
$$

and

$$
\begin{aligned}
A_{11}(\xi) &= \int_{-1}^{1} f_{11}(x) f'_{11}(x) d\xi(x) \\
A_{12}(\xi) &= \int_{-1}^{1} f_{11}(x) f'_{22}(x) d\xi(x) \\
A_{22}(\xi) &= \int_{-1}^{1} f_{22}(x) f'_{22}(x) d\xi(x)
\end{aligned}
$$

I_k is $k \times k$ identity matrix, and $\mathbf{1}$ denotes a $k \times 1$ vector with all elements 1.

Without loss of generality, we consider only designs with nonsingular information matrices. For any design ξ, denote $d_\Sigma(\xi, x) = tr[M^{-1}(\xi)F(x)\Sigma^{-1}F'(x)]$. An equivalence theorem corresponding to D-optimal criterion in multiresponse model can be found in Fedorov (1972), which states that a design ξ^D is D-optimal for multiresponse models as follows.

Theorem 3.1 If $M(\xi^D)$ is nonsingular, then following assertions are equivalent.

(1) the design ξ^D maximizes det $M(\xi)$

(2) the design ξ^D minimizes $\max_x d_\Sigma(\xi, x)$

(3) $\max_x d_\Sigma(\xi^D, x) = l$

where $l = (r+1) + k(m-r)$ is the number of the unknown parameters and the equality holds at the support points.

In Chang et al. (1999), the D-optimal designs with $(r, m) = (1, 2)$ or $(1, 3)$ for different degree of model is dependent on the ρ. In the following, it is shown that for model (3.1) the D-optimal designs are also independent of the covariance matrix.

Theorem 3.2 Consider the model (3.1). The D-optimal designs are independent of covariance matrix Σ.

proof: The determinant of information matrix $M(\xi)$ can be written as

$$\begin{aligned}
|M(\xi)| &= \begin{vmatrix} A_{11}(\xi) \otimes \Sigma^{-1} & A_{12}(\xi) \otimes (\Sigma^{-1}\mathbf{1}) \\ A_{12}'(\xi) \otimes (\mathbf{1}'\Sigma^{-1}) & A_{22}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1}) \end{vmatrix} \\
&= |A_{11}(\xi) \otimes \Sigma^{-1}||A_{22}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1}) \\
&\quad -(A_{12}'(\xi) \otimes (\mathbf{1}'\Sigma^{-1}))(A_{11}^{-1}(\xi) \otimes \Sigma)(A_{12}(\xi) \otimes (\Sigma^{-1}\mathbf{1}))| \\
&= |A_{11}(\xi) \otimes \Sigma^{-1}||A_{22}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1}) - (A_{12}'(\xi)A_{11}^{-1}(\xi)A_{12}(\xi)) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1})| \\
&= |A_{11}(\xi) \otimes \Sigma^{-1}||(A_{22}(\xi) - A_{12}'(\xi)A_{11}^{-1}(\xi)A_{12}(\xi)) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1})| \\
&= |A_{11}(\xi)|^k|\Sigma^{-1}|^{m-r}\left(\tfrac{|A(\xi)|}{|A_{11}(\xi)|}\right)(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{r+1} \\
&= |A(\xi)||A_{11}(\xi)|^{k-1}|\Sigma^{-1}|^{m-r}(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{r+1}
\end{aligned}$$

where $A(\xi) = \int_{-1}^{1} f(x)f'(x)d\xi(x)$, and $f(x) = (f_{11}'(x), f_{22}'(x))'$. The determinant of information matrix is proportional to $|A(\xi)||A_{11}(\xi)|^{k-1}$. That is ξ^D is D-optimal if and only if, ξ^D maximizes $|A(\xi)||A_{11}(\xi)|^{k-1}$. Thus the D-optimal designs are independent of Σ.

For any design ξ, let $\tilde{\xi}$ be the conjugate design of ξ, i.e., $\tilde{\xi}(-x) = \xi(x)$, for all support points x of ξ. For model (3.1), it is easy to check that $|M(\xi)| = |M(\tilde{\xi})|$ and as $\log |M(\xi)|$ is a strictly concave function in ξ, thus,

$$|M(\frac{\xi + \tilde{\xi}}{2})| \geq |M(\xi)|^{1/2} |M(\tilde{\xi})|^{1/2} = |M(\xi)|,$$

which implies that we can restrict our attention to symmetric designs.

Moreover, in model (3.1), for D-optimality, the corresponding dispersion function $d_\Sigma(\xi, x)$ is a polynomial of degree $2m$ for any design ξ. Thus, by the equivalence theorem, the D-optimal design has at most $m+1$ support points. Furthermore, the optimal design has $m+1$ support points, it must contain the two boundary points $\{-1, 1\}$.

The model under consideration here is invariant or symmetric with respect to the group consisting of permutations and sign changes of the coordinates. The invariance theorem which concludes that there exists a symmetric D-optimal designs is a very important tool for obtaining D-optimal designs either theoretically or numerically. In the following we only consider symmetric approximate designs.

Theorem 3.3 Consider a polynomial regression with k dimensional responses as defined in model (3.1) with $r = 0$ and $m = 1$. The design ξ^D is D-optimal where

$$\xi^D = \left\{ \begin{array}{cc} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{array} \right\}.$$

proof: Let $c_j = \int_{-1}^{1} x^j d\xi(x)$. In the model (3.1) with $r = 0$ and $m = 1$, the determinant of information matrix of a symmetric design ξ is

$$\begin{aligned} |M(\xi)| &\propto |A(\xi)||A_{11}(\xi)|^{k-1} \\ &= \left| \begin{array}{cc} 1 & 0 \\ 0 & c_2 \end{array} \right| \left| \begin{array}{c} 1 \end{array} \right|^{k-1} \\ &= c_2 \end{aligned}$$

where $0 \leq c_2 \leq 1$. Thus, in order to maximize $|M(\xi)|$, it is easy to see that the D-optimal design ξ^D is

$$\left\{ \begin{array}{cc} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{array} \right\}.$$

The above theorem may be applied to parallel line assay with linear regression model. In biological assays, in some circumstances the regression equation for S may be quadratic. In Theorem 3.4 the D-optimal designs for quadratic regression are discussed.

Theorem 3.4 Consider a polynomial regression with k dimensional responses as defined in model (3.1) with $r = 1$ and $m = 2$. The D-optimal design ξ^D is

$$\xi^D = \left\{ \begin{array}{ccc} -1 & 0 & 1 \\ \frac{k+1}{2(k+2)} & \frac{1}{k+2} & \frac{k+1}{2(k+2)} \end{array} \right\}.$$

proof: Let $c_j = \int_{-1}^{1} x^j d\xi(x)$. In the model (2.1) with $r = 1$ and $m = 2$, the determinant of information matrix of a symmetric design ξ is

$$\begin{aligned} |M(\xi)| &\propto |A(\xi)||A_{11}(\xi)|^{k-1} \\ &= \begin{vmatrix} 1 & 0 & c_2 \\ 0 & c_2 & 0 \\ c_2 & 0 & c_4 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & c_2 \end{vmatrix}^{k-1} \\ &= (c_2 c_4 - c_2^3) c_2^{k-1} \\ &= c_2^k (c_4 - c_2^2) \end{aligned}$$

where $c_2^2 \leq c_4 \leq c_2$ and $0 \leq c_2 \leq 1$. Thus, in order to maximize $|M(\xi)|$, it is easy to see that we need $c_4 = c_2$. Furthermore, $|M(\xi)|_{c_2 = c_4} = c_2^{k+1} - c_2^{k+2}$ attains its maximum when $c_2 = \frac{k+1}{k+2}$. Thus, the D-optimal design ξ^D is

$$\left\{ \begin{array}{ccc} -1 & 0 & 1 \\ \frac{k+1}{2(k+2)} & \frac{1}{k+2} & \frac{k+1}{2(k+2)} \end{array} \right\}.$$

Now we discussed a variation of the above model. Let P be a $l \times l$ permutation matrix, then $G(x) = PF(x)$ where $G(x) = [g_1(x), \cdots, g_k(x)]$. From this transformation, we can obtain the more general result as follows.

Consider a polynomial regression with k dimensional response variables $Y = (Y_1, Y_2, \cdots, Y_k)'$

$$\left\{ \begin{array}{lcl} E(Y_1) &=& \theta_{j_1,1} x^{j_1} + \theta_{j_2,1} x^{j_2} + \cdots + \theta_{j_r,1} x^{j_r} + \theta_{j_{r+1}} x^{j_{r+1}} + \cdots + \theta_{j_m} x^{j_m} \\ E(Y_2) &=& \theta_{j_1,2} x^{j_1} + \theta_{j_2,2} x^{j_2} + \cdots + \theta_{j_r,2} x^{j_r} + \theta_{j_{r+1}} x^{j_{r+1}} + \cdots + \theta_{j_m} x^{j_m} \\ &\vdots& \\ E(Y_k) &=& \theta_{j_1,k} x^{j_1} + \theta_{j_2,k} x^{j_2} + \cdots + \theta_{j_r,k} x^{j_r} + \theta_{j_{r+1}} x^{j_{r+1}} + \cdots + \theta_{j_m} x^{j_m} \end{array} \right. \tag{3.2}$$

where $x \in \Omega = [-1, 1]$, with $Cov(Y) = \Sigma$ and $j_i \in \aleph$, $j_p \neq j_q$, $\forall p \neq q$. Here the common parameters do not necessarily appear in the last $m-r$ terms, but can appear in any place. Then through reparameter transformation, we can always put those terms with common parameters at the end. In the following we have the similar result as Theorem 3.2.

Theorem 3.5 Consider the model (3.2). The D-optimal designs are independent of covariance matrix Σ.

proof: The determinant of information matrix $M(\xi)$ can be written as

$$\begin{aligned}
|M(\xi)| &= \begin{vmatrix} B_{11}(\xi) \otimes \Sigma^{-1} & B_{12}(\xi) \otimes (\Sigma^{-1}\mathbf{1}) \\ B'_{12}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}) & B_{22}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1}) \end{vmatrix} \\
&= |B_{11}(\xi) \otimes \Sigma^{-1}||B_{22}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1}) \\
&\quad -(B'_{12}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}))(B_{11}^{-1}(\xi) \otimes \Sigma)(B_{12}(\xi) \otimes (\Sigma^{-1}\mathbf{1}))| \\
&= |B_{11}(\xi) \otimes \Sigma^{-1}||B_{22}(\xi) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1}) - (B'_{12}(\xi)B_{11}^{-1}(\xi)B_{12}(\xi)) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1})| \\
&= |B_{11}(\xi) \otimes \Sigma^{-1}||(B_{22}(\xi) - B'_{12}(\xi)B_{11}^{-1}(\xi)B_{12}(\xi)) \otimes (\mathbf{1}'\Sigma^{-1}\mathbf{1})| \\
&= |B_{11}(\xi)|^k|\Sigma^{-1}|^{m-r}\left(\tfrac{|B(\xi)|}{|B_{11}(\xi)|}\right)(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{r+1} \\
&= |B(\xi)||B_{11}(\xi)|^{k-1}|\Sigma^{-1}|^{m-r}(\mathbf{1}'\Sigma^{-1}\mathbf{1})^{r+1}
\end{aligned}$$

where

$$\begin{aligned}
B_{11}(\xi) &= \int_{-1}^{1} g_{11}(x)g'_{11}(x)d\xi(x) \\
B_{12}(\xi) &= \int_{-1}^{1} g_{11}(x)g'_{22}(x)d\xi(x) \\
B_{22}(\xi) &= \int_{-1}^{1} g_{22}(x)g'_{22}(x)d\xi(x)
\end{aligned}$$

and $B(\xi) = \int_{-1}^{1} g(x)g'(x)d\xi(x)$, $g(x) = (g'_{11}(x), g'_{22}(x))'$, and $g'_{11}(x)$ is the vector with different parameters, $g'_{22}(x)$ is the vector with common parameters. The determinant of information matrix is proportional to $|B(\xi)||B_{11}(\xi)|^{k-1}$. That is ξ^D is D-optimal if and only if, ξ^D maximizes $|B(\xi)||B_{11}(\xi)|^{k-1}$. Thus the D-optimal designs are independent of Σ.

From Theorem 3.5, it is known that the D-optimal design is independent of the covariance structure between the k responses when the degrees of the k responses are of the same order. A simple example is given below for illustration.

Example 1

$$\begin{cases} E(Y_1) &= \beta_0 + \beta_{1,1}x + \beta_2 x^2 \\ E(Y_2) &= \beta_0 + \beta_{1,2}x + \beta_2 x^2 \end{cases},$$

where $Cov(Y_1, Y_2) = \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.
Let P and $F(x)$ be defined as

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, F(x) = \begin{pmatrix} 1 & 1 \\ x & 0 \\ 0 & x \\ x^2 & x^2 \end{pmatrix},$$

then $G(x)$ can be expressed as

$$G(x) = PF(x) = \begin{pmatrix} x & 0 \\ 0 & x \\ 1 & 1 \\ x^2 & x^2 \end{pmatrix}.$$

Therefore the D-optimal design is independent of Σ by Theorem 3.5.

In Chang et al. (1999) it considers the dual response model with some common parameters but for each response the model is different order it is found there that D-optimal design is dependent on the correlation coefficient ρ. We present a result from Chang et al. (1999) for illustration.

Example 2 Let

$$\begin{cases} E(Y_1) &= \beta_0 + \beta_{1,1}x \\ E(Y_2) &= \beta_0 + \beta_{1,2}x + \beta_{2,2}x^2 \end{cases},$$

where $Cov(Y_1, Y_2) = \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. The design ξ_ρ is D-optimal, where if

$$\rho \geq -\frac{1}{3}\ ,\ \xi_\rho = \left\{ \begin{array}{cc} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{array} \right\},$$

and if

$$\rho < -\frac{1}{3}\ ,\ \xi_\rho = \left\{ \begin{array}{ccc} -1 & 0 & 1 \\ \frac{2}{3(1-\rho)} & \frac{-1-3\rho}{3(1-\rho)} & \frac{2}{3(1-\rho)} \end{array} \right\}.$$

4 Exact D-optimal designs for k polynomial models

In this section, we discuss the exact D-optimal design of quadratic regression for parallel line assay. Consider model (3.1) with $k = 2$, $m = 2$, $r = 0$ and $r = 1$, respectively, that is considering the following two models

$$\begin{cases} E(Y_1) &= \theta_{0,1} + \theta_1 x + \theta_2 x^2 \\ E(Y_2) &= \theta_{0,2} + \theta_1 x + \theta_2 x^2 \end{cases}, \tag{4.1}$$

$$\begin{cases} E(Y_1) &= \theta_{0,1} + \theta_{1,1} x + \theta_2 x^2 \\ E(Y_2) &= \theta_{0,2} + \theta_{1,2} x + \theta_2 x^2 \end{cases}. \tag{4.2}$$

where $Cov(Y_1, Y_2) = \Sigma = (\sigma_{ij})$, $\sigma_{11} = \sigma_{22} = \sigma^2$, $\sigma_{12} = \rho\sigma_1\sigma_2$.

From Theorem 3.4, we know that the D-optimal design is independent of the covariance matrix and the corresponding approximate D-optimal design for model (4.1) and model (4.2) on $\Omega \in [-1, 1]$ are ξ_0^D and ξ_1^D, respectively, where

$$\xi_0^D = \left\{ \begin{array}{ccc} -1 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{array} \right\},$$

$$\xi_1^D = \left\{ \begin{array}{ccc} -1 & 0 & 1 \\ 3/8 & 1/4 & 3/8 \end{array} \right\},$$

Now, we consider the exact D-optimal design for model (4.1) and (4.2).

Let $x_1, \cdots, x_q$ denote the distinct levels at which $n_1, \cdots, n_q$ observations are to be taken. Here $n_1 + \cdots + n_q = n$. An exact n-point design is a probability measure ξ_n on $[-1, 1]$ where $\xi_n(x_i) = n_i/n$ denotes the relative proportion of total observations at the point x_i. The set of all exact designs for a given value of n will be denoted by E_n.

The information matrix of an exact design ξ_n is

$$M(\xi_n) = \sum_{i=1}^{q} F(x_i)\Sigma^{-1}F'(x_i)\xi_n(x_i).$$

A design ξ_n^* is called an exact n-points optimal design if ξ_n^* maximizes the determinant of $M(\xi_n)$ over the set E_n of all n-point design on $[-1, 1]$.

In the case of univariate polynomial regression, Salaevskii (1966) conjectures that an exact D-optimal design puts mass as equally as possible among the support points of the approximate D-optimal design for polynomial regression. However, the D-optimal n-point designs for the multiresponse model (4.1) and (4.2) may not always be obtained in this manner. A similar case is discussed in Krafft and Schaefer (1992). They have obtained some partial results under their model where all unknown parameters are allowed to be distinct. Later, Imhof (2000) completes the proofs for the remaining results.

It is known that the solutions ξ_n^* for exact D-optimal design problem are more difficult to obtain than that of approximate designs, because the nice convexity structure for the latter case is destroyed by restricting to exact designs. For the model (4.1), it is found that the exact D-optimal design is the same as that for a single response quadratic regression model. Exact D-optimal designs for a single response quadratic regression are due to Gaffke and Krafft (1982). But for the model (4.2), we find that the optimal designs are the same as that for the model in Krafft and Schaefer (1992). We will discuss the reasons why the exact D-optimal design are the same for the two models in the following.

From Theorem 3.2, the determinant of information matrix is proportional to $|A(\xi)||A_{11}(\xi)|^{k-1}$. For model (4.2),

$$|A(\xi)||A_{11}(\xi)| = \begin{vmatrix} n & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} \begin{vmatrix} n & s_1 \\ s_1 & s_2 \end{vmatrix}$$

where $x \in [-1, 1]^n$

$$s_i = s_i(x) = \sum_{k=1}^{n} x_k^i, \ 1 \leq i \leq 4.$$

This form is exactly the same function to be maximized, therefore exact D-optimal design is the same as the model that Krafft and Schaefer (1992) and the proof is given in Appendix. In the general case with k responses, we have shown the D-optimal design is still independent of the covariance structure. Therefore we only need to discuss the case with independence among the k responses. In the following, through some row and column operations, it is shown that the problem may be reduced to another form. This results may also be applied to the case studied in Krafft and Schaefer (1992) with $k = 2$.

For case of illustration, we consider a simple case $k = 2$ for model (3.1), by row and column operations

$$\begin{aligned}
\sum_{i=1}^{n} \begin{bmatrix} f_1 & 0 \\ 0 & f_1 \\ f_2 & f_2 \end{bmatrix} \begin{bmatrix} f_1' & 0 & f_2' \\ 0 & f_1' & f_2' \end{bmatrix} &= \sum_{i=1}^{n} \begin{bmatrix} f_1 f_1' & 0 & f_1 f_2' \\ 0 & f_1 f_1' & f_1 f_2' \\ f_2 f_1' & f_2 f_1' & 2 f_2 f_2' \end{bmatrix} \\
&\to \sum_{i=1}^{n} \begin{bmatrix} f_1 f_1' & 0 & f_1 f_2' \\ -f_1 f_1' & f_1 f_1' & f_1 f_2' \\ 0 & f_2 f_1' & 2 f_2 f_2' \end{bmatrix} \\
&\to \sum_{i=1}^{n} \begin{bmatrix} f_1 f_1' & 0 & f_1 f_2' \\ 0 & f_1 f_1' & 2 f_1 f_2' \\ 0 & f_2 f_1' & 2 f_2 f_2' \end{bmatrix}
\end{aligned}$$

Then the problem turns to determine

$$\sup\{|\sum_{i=1}^{n} f_1 f_1'| \begin{vmatrix} \sum_{i=1}^{n} f_1 f_1' & \sum_{i=1}^{n} f_1 f_2' \\ \sum_{i=1}^{n} f_2 f_1' & \sum_{i=1}^{n} f_2 f_2' \end{vmatrix} : x \in [-1, 1]^n\}, \tag{4.3}$$

where $f_1 = f_1(x) = (1, x, \cdots, x^r)'$, $f_2 = f_2(x) = (x^{r+1}, \cdots, x^m)'$.

In the case with $k \geq 2$, it can be seen that

$$\begin{aligned}
\sum_{i=1}^{n} \begin{bmatrix} f_1 \otimes I_k \\ f_2 \otimes 1' \end{bmatrix} \begin{bmatrix} f_1' \otimes I_k & f_2' \otimes 1 \end{bmatrix} &= \sum_{i=1}^{n} \begin{bmatrix} f_1 f_1' \otimes I_k & f_1 f_2' \otimes 1 \\ f_2 f_1' \otimes 1' & f_2 f_2' \otimes k \end{bmatrix} \\
&\to \sum_{i=1}^{n} \begin{bmatrix} f_1 f_1' \otimes I_{k-1} & 0 & f_1 f_2' \otimes 1 \\ 0 & f_1 f_1' & k f_1 f_2' \\ 0 & f_2 f_1' & k f_2 f_2' \end{bmatrix}
\end{aligned}$$

Therefore the problem turns to determine

$$\sup\{|\sum_{i=1}^{n} f_1 f_1'|^{k-1} \begin{vmatrix} \sum_{i=1}^{n} f_1 f_1' & \sum_{i=1}^{n} f_1 f_2' \\ \sum_{i=1}^{n} f_2 f_1' & \sum_{i=1}^{n} f_2 f_2' \end{vmatrix} : x \in [-1, 1]^n\}, \tag{4.4}$$

Finally, we extend the more general result for the multiresponse model with some common parameters, but the positions of the common parameters are not restricted to the last terms. More explicitly consider a polynomial regression with k dimensional response variables $Y = (Y_1, Y_2, \cdots, Y_k)'$

$$\begin{cases} E(Y_1) &= \theta_{j_1,1}x^{j_1} + \theta_{j_2,1}x^{j_2} + \cdots + \theta_{j_r,1}x^{j_r} \\ E(Y_2) &= \theta_{j_1,2}x^{j_1} + \theta_{j_2,2}x^{j_2} + \cdots + \theta_{j_r,2}x^{j_r} \\ &\vdots \\ E(Y_k) &= \theta_{j_1,k}x^{j_1} + \theta_{j_2,k}x^{j_2} + \cdots + \theta_{j_r,k}x^{j_r} + \theta_{j_{r+1}}x^{j_{r+1}} + \cdots + \theta_{j_m}x^{j_m} \end{cases} \tag{4.5}$$

where $x \in \Omega = [-1, 1]$, with $Cov(Y) = \Sigma$ and $j_i \in \aleph$, $j_p \neq j_q$, $\forall p \neq q$. Again through some matrix permutation, we have the following results.

Theorem 4.1 Consider the model (3.2). Then exists a permutation matrix, such that the exact D-optimal is the same as that for the model (4.5).

proof: Let P be a $l \times l$ permutation matrix, that is $P'P = I$, where l is the number of unknown parameters.

$$\begin{aligned} \sum_{i=1}^{n} PF(x)F'(x)P' &= \sum_{i=1}^{n} \begin{bmatrix} g_1 \otimes I_k \\ g_2 \otimes 1' \end{bmatrix} \begin{bmatrix} g_1' \otimes I_k & g_2' \otimes 1 \end{bmatrix} \\ &\to \sum_{i=1}^{n} \begin{bmatrix} g_1g_1' \otimes I_{k-1} & 0 & g_1g_2' \otimes 1 \\ 0 & g_1g_1' & kg_1g_2' \\ 0 & g_2g_1' & kg_2g_2' \end{bmatrix} \end{aligned}$$

where $g_1 = g_1(x)$ is the vector with no common parameters and $g_2 = g_2(x)$ is the vector with common parameters.
The problem is to determine

$$\sup\{|\sum_{i=1}^{n} g_1g_1'|^{k-1} \begin{vmatrix} \sum_{i=1}^{n} g_1g_1' & \sum_{i=1}^{n} g_1g_2' \\ \sum_{i=1}^{n} g_2g_1' & \sum_{i=1}^{n} g_2g_2' \end{vmatrix} : x \in [-1, 1]^n\}, \tag{4.6}$$

From (4.3), (4.4) and (4.6), we obtain an important result for any multiresponse model with some common parameters. The exact D-optimal design for model of the form as model (3.2), we only need to consider the part with distinct parameters. In other words, the part of the common parameters do not lead to any change on the results for the exact D-optimal design. Therefore, when the models are with some common parameters, just consider one of them and let the other parameters be equal to 0. We give simple example to assist us to understand.

Example 3 Consider the model in Example 1, then the exact D-optimal is the same as that of the following model.

$$\begin{cases} E(Y_1) &= \beta_{1,1}x \\ E(Y_2) &= \beta_0 + \beta_{1,2}x + \beta_2 x^2 \end{cases}.$$

In biological assay, sometimes it may use several test preparations at the same time. So it is reasonable to consider the more general symmetric cases for $k \geq 2$. We have the following theorem.

Theorem 4.2 Consider model (3.1) with $r = 1$, $m = 2$, $n \in \aleph$, for $k \geq 2$. For j in the symmetric case of Table 4.1, where

$$\begin{cases} n = (k+2)p + j, & \text{if } k \text{ is odd,} \\ n = 2(k+2)p + j, & \text{if } k \text{ is even.} \end{cases}$$

Then the support points of the exact D-optimal design ξ_n^* are $\{-1, 0, 1\}$ and

$$\begin{aligned} e_{-1} &= e_1 = \begin{cases} \frac{1}{2}[\frac{k+1}{k+2}n], & \text{if } [\frac{k+1}{k+2}j] \text{ is even,} \\ \frac{1}{2}([\frac{k+1}{k+2}n] + 1), & \text{if } [\frac{k+1}{k+2}j] \text{ is odd,} \end{cases} \\ e_0 &= n - 2e_1, \end{aligned}$$

where e_i, $i \in \{-1, 0, 1\}$ denotes the number of observations on $\{-1, 0, 1\}$, respectively.

Table 4.1 Exact D-optimal designs for k polynomial models with $r = 1$ and $m = 2$.

k	n	symmetric case j	asymmetric case j
2	$8p + j$	$0, \pm 2, \pm 3$	$\pm 1, \pm 4$
3	$5p + j$	$0, \pm 2$	± 1
4	$12p + j$	$0, \pm 2, \pm 3, \pm 5$	$\pm 1, \pm 4, \pm 6$
5	$7p + j$	$0, \pm 2, \pm 3$	± 1
6	$16p + j$	$0, \pm 2, \pm 4, \pm 5, \pm 7$	$\pm 1, \pm 3, \pm 6, \pm 8$
7	$9p + j$	$0, \pm 2, \pm 4$	$\pm 1, \pm 3$
8	$20p + j$	$0, \pm 2, \pm 4, \pm 5, \pm 7, \pm 9$	$\pm 1, \pm 3, \pm 6, \pm 8, \pm 10$
9	$11p + j$	$0, \pm 2, \pm 4, \pm 5$	$\pm 1, \pm 3$

proof: Let for $x \in [-1, 1]^n$

$$s_i = s_i(x) = \sum_{k=1}^{n} x_k^i, \ 1 \le i \le 4.$$

The problem is to determine

$$\sup\{S'(x) : x \in [-1, 1]^n\},$$

where

$$\begin{aligned} S'(x) &= |M(\xi_n)||M_{11}(\xi_n)|^{k-1} \\ &= \begin{vmatrix} n & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} \begin{vmatrix} n & s_1 \\ s_1 & s_2 \end{vmatrix}^{k-1} \end{aligned}$$

We will show that $S'(x)$ has an upper bound on $[-1, 1]$ which is attained when $x \in \{-1, 0, 1\}^n$. First we have

$$\begin{aligned} S'(x) &= (ns_2s_4 + 2s_1s_2s_3 - s_2^3 - ns_3^2 - s_1^2s_4)(ns_2 - s_1^2)^{k-1} \\ &\le n^{k-1}s_2^k(ns_4 - s_2^2) \\ &= n^{k-1}(\textstyle\sum_{i=1}^n y_i)^k(n\sum_{i=1}^n y_i^2 - (\sum_{i=1}^n y_i)^2) = W(y) \end{aligned}$$

where $x_i^2 = y_i$ and $W(y)$ attains its supremum on $[0, 1]^n$ in the vertices of $[0, 1]^n$. Considering thus $W(y)$ on $\{0, 1\}^n$, let

$$w = w(y) = |\{i : y_i = 1\}|.$$

Then

$$W(y) = n^{k-1} w^k (nw - w^2).$$

We can see that $W(y)$ is maximal iff

$$\begin{array}{ll} w = [\frac{k+1}{k+2} n], & \text{if } [\frac{k+1}{k+2} j] \text{ is even,} \\ w = [\frac{k+1}{k+2} n] + 1, & \text{if } [\frac{k+1}{k+2} j] \text{ is odd,} \end{array}$$

where $[z]$ denotes the greatest integer less than or equal to z.

But if the $(k+1)j \ (mod\ (k+2)) \equiv (k+2)/2$, for example, $k = 2$ and $n = 8p + 2$, it can not be proved by this method. From numerical results we discover that for $k = 5$, $p = 1, 2$, the structure of the exact D-optimal design will change. Fortunately, for large enough n the structure will be consistent.

5 Discussion

In this work we discuss the approximate and exact D-optimal designs for multiresponse polynomial regression models with common parameters motivated from parallel line assays. First, the approximate D-optimal designs are shown to be independent of the covariance structure between the k responses when the degrees of the k responses are of the same order and with common parameters at the last few terms of the polynomial. Then we extend this result further in Theorem 3.5.

The exact D-optimal design problem of the model (4.1) for any k, it is found that the exact D-optimal design is the same as a single response with quadratic regression model. Then in model (4.2), we also detect the structure of the exact D-optimal design is the same as that from Krafft and Schaefer (1992) and Imhof (2000). Later, in Theorem 4.1 we provide a model with more general structure to illustrate results under the structure of model (4.1) and (4.2).

Besides, we extend general results for the symmetric cases with $k \geq 2$. In Imhof (2000) the problem for $n = 8p + 2$ and the other asymmetric cases are solved, but the techniques used there are very complicate, we are unable to extend the method there for the general k right now. From numerical results, we detect that in some small samples the structure of the exact D-optimal designs may change. For example, when $k = 5$, $p = 1, 2$, $j = 4$ the exact D-optimal design is asymmetric and for $p \geq 3$ they are symmetric. Although for large samples the structure is fixed. It is quite possible the situation is also like in Salaevskii conjecture that a fixed pattern exists only for sample size large enough. In the future work, we may discuss the asymmetric cases for general k.

6 Appendix

Krafft and Schaefer (1992) consider the linear and quadratic regression model that

$$\begin{cases} EY_i(x) = \theta_0^{(i)} + \theta_1^{(i)} x & i = 1, \cdots, m \\ EY_i(x) = \theta_0^{(i)} + \theta_1^{(i)} x + \theta_2^{(i)} x^2 & i = m+1, \cdots, 2m. \end{cases} \tag{6.1}$$

where $\theta_0^{(i)}, \theta_1^{(i)}, \theta_2^{(i)}$ are unknown regression parameters. Assume the covariance matrix $Cov(Y_1(x), \cdots, Y_{2m}(x)) = \Sigma$ is independent of x.

Theorem 6.1 (Krafft and Schaefer (1992)) Let $n \in \aleph$, $n = 8p + j$, $j \in \{0, 3, 5, 6\}$. Then for model (6.1) the support points of exact D-optimal design ξ_n^* are $\{-1, 0, 1\}$ and

$$\begin{aligned} e_{-1} &= e_1 = \begin{cases} 3p, & \text{if } j = 0, \\ 3p+1, & \text{if } j = 3, \\ 3p+2, & \text{if } j = 5 \text{ or } j = 6, \end{cases} \\ e_0 &= n - 2e_1, \end{aligned}$$

where e_i, $i \in \{-1, 0, 1\}$ denotes the observations of x_i.

proof: Let for $x \in [-1, 1]^n$

$$s_i = s_i(x) = \sum_{k=1}^{n} x_k^i, \ 1 \leq i \leq 4.$$

The problem is to determine

$$\sup\{S(x) : x \in [-1, 1]^n\},$$

where

$$\begin{aligned} S(x) &= |M(\xi_n)||M_{11}(\xi_n)| \\ &= \begin{vmatrix} n & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} \begin{vmatrix} n & s_1 \\ s_1 & s_2 \end{vmatrix} \end{aligned}$$

We will show that $S(x)$ has an upper bound on $[-1,1]$ which is attained when $x \in \{-1,0,1\}^n$. First we have

$$\begin{array}{rcl} S(x) & = & (ns_2s_4 + 2s_1s_2s_3 - s_2^3 - ns_3^2 - s_1^2s_4)(ns_2 - s_1^2) \\ & \leq & ns_2^2(ns_4 - s_2^2) \\ & = & n(\sum_{i=1}^n y_i)^2(n\sum_{i=1}^n y_i^2 - (\sum_{i=1}^n y_i)^2) = V(y) \end{array}$$

where $x_i^2 = y_i$($V(y)$ attains its supremum on $[0,1]^n$ in the vertices of $[0,1]^n$). Considering thus $V(y)$ on $\{0,1\}^n$, let

$$v = v(y) = |\{i : y_i = 1\}|.$$

Then

$$V(y) = nv^2(nv - v^2).$$

We can see that $V(y)$ is maximal iff

$$\begin{array}{ll} v = [\frac{3}{4}n], & \text{if } j = 0, 3, \text{or } 6, \\ v = [\frac{3}{4}n] + 1, & \text{if } j = 5, \end{array}$$

where $[z]$ denotes the greatest integer less than or equal to z.

From the above proof, the $n = 8p + 2$ is a special case which can not be treated similarly as the other symmetric case. We must use different method to solve it. Imhof (2000) gives the proofs for the following Theorem 6.2 to Theorem 6.5.

Theorem 6.2 Let $n \in \aleph$, $n = 8p + 2$. Then for model (6.1) the support points of exact D-optimal design ξ_n^* are $\{-1, 0, 1\}$ and

$$\left\{ \begin{array}{ccc} -1 & 0 & 1 \\ 3p+1 & 2p & 3p+1 \end{array} \right\}.$$

The proofs for the asymmetric case is more complicate than that for symmetric case. Therefore, we consider that all support points in $[-1,1]^n$.

Theorem 6.3 Let $n \in \aleph$, $n = 8p + 1$ and x_0 be the real root of $w(x) = 0$, where

$$w(x) = (9n+3)x^3 - 20x^2 + (21n+31)x + 4$$

Then for model (6.1) the support points of exact D-optimal design ξ_n^* are

$$\left\{ \begin{array}{ccc} -1 & -x_0 & 1 \\ 3p+1 & 2p & 3p \end{array} \right\} \cup \left\{ \begin{array}{ccc} -1 & x_0 & 1 \\ 3p & 2p & 3p+1 \end{array} \right\}.$$

Theorem 6.4 Let $n \in \aleph$, $n = 8p + 4$ and x_0 be the real root of $w(x) = 0$, where

$$w(x) = 9n^2x^3 - 20nx^2 + (21n^2 - 32)x + 4n$$

Then for model (6.1) the support points of exact D-optimal design ξ_n^* are

$$\left\{ \begin{array}{ccc} -1 & -x_0 & 1 \\ 3p+2 & 2p & 3p+1 \end{array} \right\} \cup \left\{ \begin{array}{ccc} -1 & x_0 & 1 \\ 3p+1 & 2p & 3p+2 \end{array} \right\}.$$

Theorem 6.5 Let $n \in \aleph$, $n = 8p + 7$ and x_0 be the real root of $w(x) = 0$, where

$$w(x) = (9n - 3)x^3 - 20x^2 + (21n - 31)x + 4$$

Then for model (6.1) the support points of exact D-optimal design ξ_n^* are

$$\left\{ \begin{array}{ccc} -1 & -x_0 & 1 \\ 3p+3 & 2p+2 & 3p+2 \end{array} \right\} \cup \left\{ \begin{array}{ccc} -1 & x_0 & 1 \\ 3p+2 & 2p+2 & 3p+3 \end{array} \right\}.$$

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