Contents

Abstract

In this work, we investigate c-optimal design for polynomial regression model without intercept. Huang and Chen (1996) showed that the c-optimal design for the d^{th} degree polynomial with intercept is still the optimal design for the no-intercept model for estimating certain individual coefficients over $[-1, 1]$. We found the c-optimal designs explicitly for estimating other individual coefficients over $[-1, 1]$, which have not been obtained earlier. For the no-intercept model, it is shown that the support points are scale invariant over $[-b, b]$. Finally some special cases are discussed for estimating the coefficients of the 2^{nd} degree polynomial without intercept by Elfving theorem over nonsymmetric interval $[a, b]$.

Keywords : c-optimal design, Elfving Theorem, individual regression coefficient.

C**-optimal designs for polynomial regression without intercept.**

1 Introduction

Let $E(y(x)) = \theta' f(x)$ be the homoscedasic regression model on $[a, b]$, where $f'(x) = (f_0(x), f_1(x), \cdots, f_d(x))$ denote a known vector with $d + 1$ linearly independent continuous functions on $[a, b]$, and $\theta' = (\theta_0, \theta_1, \cdots, \theta_d)$ is the unknown parameter vector. The outcome variable $y(x)$ with mean value $\sum_{i=0}^{d} \theta_i f_i(x)$ and a common variance σ^2. Let $Y = [y(x_1), \cdots, y(x_n)]'$, $X = [f(x_1), \cdots, f(x_n)]'$, where $y(x_1), \cdots, y(x_n)$ are the uncorrelated random responses. Least squares estimator of the unknown parameter vector θ is $\hat{\theta} = (X'X)^{-1}X'Y$. Then $E(\hat{\theta}) = \theta$ and $Cov(\hat{\theta}) = \sigma^2(X'X)^{-1}$.

An exact design specifies a probability measure ξ on $[a, b]$ which concentrates weight p_ν at distinct x_ν and where np_ν is an integer, $\nu = 1, \cdots, n$. An approximate design is one where the integral constraint on all the np_ν is not imposed. The information matrix of a design ξ is defined by $M(\xi) = \int f(x)f'(x)d\xi(x) = (m_{ij})_{i,j=0}^{d}$, where $m_{ij} = \int f_i(x)f_j(x)d\xi(x)$, such that $M(\xi) = \frac{1}{n}(X'X)$.

Our primary goal here is to estimate a linear form $c'\theta = \Sigma_{\nu=0}^{d} c_\nu \theta_\nu$ and assume that $\Sigma_{\nu=0}^{d} c_\nu^2 > 0$, where the coefficient vector $c = (c_0, c_1, \cdots, c_d)' \in R^{d+1}$.

A linear form $c'\theta$ is called estimable with respect to ξ if c is contained in the range of the matrix $M(\xi)$. If $c'\theta$ is estimable with a design ξ, then the variance of $c'\theta$ is proportion to $c'M^{-1}(\xi)c$, where $M^{-1}(\xi)$ is a generalized inverse of the information matrix $M(\xi)$.The design is called c-optimal if minimizes $c'M^{-1}(\xi)c$ among all the design ξ on $[a, b]$ where c lies in the range of $M(\xi)$.

Hoel and Levine (1964) showed that if $f_\nu(x) = x^\nu$, $\nu = 0, 1, \cdots, d$ on $[-1, 1]$, and

$c = f(x_0)$ with $|x_0| > 1$ then the c-optimal design is supported on the Chebyshev points $s_\nu = \cos(\frac{\nu\pi}{d})$, $\nu = 0, 1, \cdots, d$. These are the points which satisfy $|T_d(x)| = 1$, where $T_d(x)$ is the d^{th} Chebyshev polynomial of the 1^{st} kind.

Kiefer and Wolfowitz (1965) considered more general systems of regression functions and a large class of designs supported by the Chebyshev points was characterized.

Studden (1968) gave a different proof of the Kiefer-Wolfowitz results and pointed out that for some special vectors c, its c-optimal design is supported by the Chebyshev points.

Chang and Heiligers (1996) found the E-optimal designs for the no-intercept regression model on symmetric interval $[-1, 1]$.

Huang and Chen (1996) showed that the c-optimal design for the d^{th} degree polynomial with intercept is still the optimal design for the no-intercept model for estimating certain individual coefficients over $[-1, 1]$. We found the c-optimal designs explicitly for estimating other individual coefficients over $[-1, 1]$, which have not been obtained earlier. For the no-intercept model, it is shown that the support points are scale invariant over $[-b, b]$. Finally some special cases are discussed for estimating the coefficients of the 2^{nd} degree polynomial without intercept by Elfving theorem over nonsymmetric interval $[a, b]$.

2 Preliminary for characterization of c-optimal designs and related results.

The following result due to Elfving (1952) characterizes the c-optimal designs ξ and will be employed throughout the paper. First we define the Elfving set R, which is the convex hull of the regression range $R_+ = \{f(x) = (f_0(x), \cdots, f_d(x))|x \in \chi\}$ and its negative image $R_- = \{-f(x)|x \in \chi\}$. That is

$$R = conv(R_+ \cup R_-).$$

Every vector $c \in R$ can be put in the form

$$c = \sum_{\nu=1}^{k} \epsilon_\nu p_\nu f(x_\nu)$$

where $\epsilon_\nu = \pm 1$, $p_\nu > 0$ and $\sum_{\nu=1}^{k} p_\nu = 1$. The integer k may always be taken to be at most $d+2$ and at most $d+1$ if c is a boundary point of R.

In the following we state the Elfving theorem and a lemma given in Studden (1968).

Theorem 2.1. *A design ξ_0 is c-optimum if and only if there exists a measurable function $\varphi(x)$ satisfying $|\varphi(x)| \equiv 1$ such that*

(i) $\int \varphi(x) f(x) \xi(dx) = \beta c$ for some β

(ii) βc is a boundary point of the Elfving set R.

Lemma 2.1. *A vector c of the form*

$$c = \sum_{\nu=1}^{k} \epsilon_\nu p_\nu f(x_\nu)$$

lies on the boundary of R if and only if there exists a nontrivial "polynomial"

$$u(x) = \sum_{\nu} a_\nu f_\nu(x)$$

such that $|u(x)| \le 1$ *for* $x \in [-1,1]$, $\epsilon_\nu u(x_\nu) = 1$, $\nu = 1, 2, \cdots, k$, *and* $\sum_\nu a_\nu c_\nu = 1$.

Studden(1968) showed that if $f_0, \cdots, f_d$ form a Chebyshev system over $[-1,1]$ then there are two classes of vectors c, where the support points $s_0, \cdots, s_d$, of the c-optimal design are the alternating extreme points of the unique Chebyshev polynomial.

The classes of vectors c can be determined by the sign pattern of the determinants

$$\begin{aligned} D_\nu(c) &= \begin{vmatrix} f_0(s_0) & \cdots & f_0(s_{\nu-1}) & f_0(s_{\nu+1}) & \cdots & f_0(s_d) & c_0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ f_d(s_0) & \cdots & f_d(s_{\nu-1}) & f_d(s_{\nu+1}) & \cdots & f_d(s_d) & c_d \end{vmatrix} \\ &= |f(s_0) \cdots f(s_{\nu-1}) f(s_{\nu+1}) \cdots f(s_d)\ c| \end{aligned}$$

The sign of $D_\nu(c)$ will be denoted by $d_\nu(c)$; if $D_\nu(c) = 0$ the sign may be defined as -1 or $+1$.

Then we introduce the c-optimal Weights Theorem on linearly independent regression vectors that provides explicit formula for calculation the c-optimal weights. The theorem is from Pukelsheim and Torsney (1991), and has forerunners in Studden (1971, Theorem 3.1), and Kitsos, Titterington and Torsney (1988, section 6.1).

Theorem 2.2. *There exist linearly independent regression vectors* $x_1, x_2, \cdots, x_k$ *in* χ *that support an optimal design* ξ *for* $c'\theta$. *The weights* $w_\nu = \xi(x_\nu)$, *satisfy*

$$w_\nu = \frac{|u_\nu|}{\Sigma_{j \le k} |u_j|} \ for\ all\ \nu = 1, 2, \cdots, k$$

where $u_1, u_2, \cdots, u_k$, *are the components of the vector* $u = (XX')^{-1} Xc$, *and*

$$X' = (f(x_1), f(x_2), \cdots, f(x_k))$$

be a $k \times k$ nonsingular matrix.

In c-optimality the interest is in estimating the linear combination of the parameters $c'\theta$ with minimun variance. The design criterion to be minimized is thus

$$Var(c'\theta) \propto c'M^{-1}(\xi)c \quad .$$

The equivalence theorem states that, for the optimum design,

$$\{c'M^{-1}(\xi^*)f(x)\}^2 \le c'M^{-1}(\xi^*)c$$

for $x \in \chi$.

We define some notations which are useful for later discussions. If there are d distinct points $a \le x_0 < \cdots < x_{d-1} \le b$ such that $f(x_i) \ne 0$ for all $0, 1, \cdots, d-1$, let

$$F\begin{pmatrix} 0 & 1 & \cdots & d-1 \\ x_0 & x_1 & \cdots & x_{d-1} \end{pmatrix} = |f(x_0)f(x_1)\cdots f(x_{d-1})| \ne 0 \ ,$$

and

$$L_\nu(x) = F\begin{pmatrix} 0 & 1 & \cdots & \nu-1 & \nu & \nu+1 & \cdots & d-1 \\ x_0 & x_1 & \cdots & x_{\nu-1} & x & x_{\nu+1} & \cdots & x_{d-1} \end{pmatrix} / F\begin{pmatrix} 0 & 1 & \cdots & d-1 \\ x_0 & x_1 & \cdots & x_{d-1} \end{pmatrix}$$

denote the Lagrange interpolation polynomial defined by requiring that $L_\nu(x_j) = \delta_{\nu j}$, $\nu, j = 0, \cdots, d-1$.

Next we state a formulas for finding the inverse of a nonsingular Vandermonde matrix adopted from Graybill (1983 p.265-270), which will also be used later.

Let $x_0, x_1, \cdots, x_{d-1}$ be a set of d real numbers and let X be a $d \times d$ Vandermonde matrix defined by

$$X = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{d-1} \\ x_0^2 & x_1^2 & \cdots & x_{d-1}^2 \\ \vdots & \vdots & & \vdots \\ x_0^{d-1} & x_1^{d-1} & \cdots & x_{d-1}^{d-1} \end{pmatrix} \tag{1}$$

and define the $(d-1)^{th}$ degree polynomial $P_\nu(x)$ by

$$\begin{aligned} P_\nu(x) &= (x-x_0)(x-x_1)\cdots(x-x_{\nu-1})(x-x_{\nu+1})\cdots(x-x_{d-1}) \\ &= \prod_{\substack{t=0 \\ t\neq\nu}}^{d-1} (x-x_t) \quad for \;\; \nu = 0,1,\cdots,d-1. \end{aligned} \tag{2}$$

If we multiply the factors in $P_\nu(x)$, we get

$$P_\nu(x) = \sum_{j=0}^{d-1} a_{\nu j} x^j \quad for \;\; \nu = 0,1,\cdots,d-1, \tag{3}$$

where $\{a_{\nu j}\}$ are the appropriate products and sums of the numbers $x_0, x_1, \cdots, x_{d-1}$.

We now state a theorem for the inverse of X.

Theorem 2.3. *Let X be a $d \times d$ Vandermonde matrix given in (1), where the x_ν are distinct. Denote X^{-1} by $B = [b_{mn}]$. The elements of b_{mn} are*

$$b_{mn} = \frac{a_{\nu j}}{P_\nu(x_\nu)} \quad , \tag{4}$$

where $m = \nu + 1$, $n = j + 1$, and $\nu, j = 0, 1, \cdots, d-1$.

3 Optimal designs for the individual regression coefficients

3.1 Individual regression coefficients for the no-intercept model over [-1,1]

For the no-intercept model $f(x) = (x, \cdots, x^d)'$ on $[-1, 1]$, this model is the same as the weighted polynomial model with weight function $w(x) = x^2$ and let $\tilde{f}_{d-1}(x) = (1, x, \cdots, x^{d-1})'$.

Our goal is to find the optimal design for each parameter θ_i for the weighted polynomial models on $[-1, 1]$. Let $c_p = (0, \cdots, 0, 1, 0, \cdots, 0)'$ be the vector in R^d with a one in the p^{th} component and zero elsewhere. Then $c_p'\theta = \theta_p$, for all $p = 1, \cdots, d$, with $\theta = (\theta_1, \cdots, \theta_d)'$. We divide the discussion into four cases according to d and $d - p$ are even or odd.

First we consider the case that $d - p$ is even and d is odd. Huang and Chen (1996) showed that the c_p-optimal design for the d^{th} degree polynomial model with intercept is still the optimal design for the no-intercept model for certain p's, this is given in Theorem 3.1.

Theorem 3.1. *Let $s_\nu = cos(\frac{\nu\pi}{d})$, $\nu = 0, 1, \cdots, d$ be the Chebyshev points and $\tau = (\tau_0, \tau_1, \cdots, \tau_d)'$ be the coefficient vector of the d^{th} Chebyshev polynomial of the 1^{st} kind, $T_d(x)$. For the no-intercept model on $[-1, 1]$ with odd degree d and even $d - p$, the c_p-optimal design is supported by the Chebyshev points $\{s_\nu\}$.*

For even degree d, Chang and Heiligers (1996) showed that for the no-intercept model on $[-1, 1]$, the E-optimal design for the d^{th} degree polynomial model with intercept over $[-1, 1]$ remains optimal.

They also proved that there exists an equioscillating weighted polynomial of degree d over $[-1,1]$ (weighted by $\sqrt{w}=x$)

$$C_{d-1}=\sqrt{w}c'_{d-1}\tilde{f}_{d-1} \quad ,$$

that is for a special vector $c_{d-1}=(\gamma_0,\cdots,\gamma_{d-1})'\in R^d$ will be given later, where $||C_{d-1}||=max_{x\in[-1,1]}|C_{d-1}(x)|=1$. There are d alternation points $1\geq\varrho_0>\cdots>\varrho_{d-1}\geq -1$ in $[-1,1]$ such that

$$C_{d-1}(\varrho_\nu)=(-1)^\nu||C_{d-1}|| \ , \ \ 0\leq\nu\leq d-1 \quad .$$

Actually, there is exactly one normalized weighted equioscillating polynomial C_{d-1} of degree d, The leading coefficient γ_{d-1} of C_{d-1} is different from zero, as is implied by the oscillating property and $C_{d-1}(0)=0$. The equioscillating property of C_{d-1} carries over to $C_{d-1}(x)=-C_{d-1}(-x)$, $[-1,1]$, thus C_{d-1} is an odd polynomial.

Let $l=\dfrac{d-2}{2}$ and $T_{l+1}(x)=\sum_{\nu=0}^{l+1}\tau_\nu x^\nu$, $x\in[-1,1]$, which is the Chebyshev polynomial of the first kind of degree $l+1$(normalized so that $T_{l+1}(1)=1$) with corresponding alternation points $\rho_\nu=\cos(\dfrac{\nu\pi}{l+1})$, $0\leq\nu\leq l+1$,and denote $\sigma=\cos(\dfrac{2l+1}{2l+2}\pi)$ which is the smallest zero of T_{l+1} in $[-1,1]$. Then we have the following lemma from Chang and Heiligers(1996).

Lemma 3.1. *$C_{d-1}(x)=sgn(x)T_{l+1}((1-\sigma)x^2+\sigma)$, $x\in[-1,1]$, i.e.,*

$$\gamma_\nu=\begin{cases} 0 & if\ \nu\ is\ even \\ (\frac{1-\sigma}{\sigma})^\nu\sum_{\mu=\nu}^{l+1}\binom{\mu}{\nu}\sigma^\mu\tau_\mu & if\ \nu\ is\ odd\end{cases}$$

and

$$\varrho_\nu=\begin{cases} (\frac{\rho_\nu-\sigma}{1-\sigma})^{\frac{1}{2}} & if\ 0\leq\nu\leq l \\ -(\frac{\rho_{d-1-\nu}-\sigma}{1-\sigma})^{\frac{1}{2}} & if\ l<\nu\leq d-1\end{cases}$$

Then we can find a symmetric polynomial of degree d over $[-1,1]$,

$$u_d(x) = \sqrt{w} v_d' \tilde{f}_{d-1} \quad , \quad v_d = (\alpha_0, \cdots, \alpha_{d-1})' \in R^d \quad .$$

The leading coefficient α_{d-1} of $u_d(x)$ is different from zero, and

$$\alpha_{d-1} = sgn(x)(-1)^{\frac{p}{2}} \gamma_{d-1} \quad ,$$

where p means the one in the p^{th} component of $c_p = (0, \cdots, 0, 1, 0, \cdots, 0)'$ and zero elsewhere. The symmetric property of $u_d(x)$ carries over to $u_d(x) = u_d(-x)$, $x \in [-1,1]$, thus u_d is an even polynomial. Then we can have a similar lemma as Lemma 3.1.

Let $-1 \leq x_0 < \cdots < x_{d-1} \leq 1$ be the d alternation points of $u_d(x)$ over $[-1,1]$, and $||u_d|| = max_{x \in [-1,1]} |u_d(x)| = 1$.

Lemma 3.2.

$$u_d(x) = (-1)^{\frac{p}{2}} T_{l+1}((1-\sigma)x^2 + \sigma) \quad , \quad x \in [-1,1]$$

$$\alpha_\nu = sgn(x)(-1)^{\frac{p}{2}} \gamma_\nu \quad , \quad \nu = 0, 1, \cdots, d-1 \; .$$

$$x_\nu = \begin{cases} -(\frac{\rho_{d-1-\nu} - \sigma}{1-\sigma})^{\frac{1}{2}} & if \;\; 0 \leq \nu \leq l \\ (\frac{\rho_\nu - \sigma}{1-\sigma})^{\frac{1}{2}} & if \;\; l < \nu \leq d-1 \end{cases}$$

and

$$u_d(x_\nu) = \begin{cases} (-1)^{\frac{p}{2}+\nu} & if \;\; 0 \leq \nu \leq l \\ (-1)^{\frac{p}{2}+\nu-1} & if \;\; l < \nu \leq d-1 \end{cases}$$

Proof. Since $u_d(x) = (-1)^{\frac{p}{2}} sgn(x) C_{d-1}(x)$, and

$$C_{d-1}(x_\nu) = (-1)^{\nu+1} ||C_{d-1}|| = (-1)^{\nu+1} \; , \; 0 \leq \nu \leq d-1$$

, such that it is easy to know

$$
\begin{aligned}
u_d(x_\nu) &= (-1)^{\frac{p}{2}} sgn(x)(-1)^{\nu+1} \\
&= \begin{cases} (-1)^{\frac{p}{2}+\nu} & if \ \ 0 \le \nu \le l \\ (-1)^{\frac{p}{2}+\nu-1} & if \ \ l < \nu \le d-1 \end{cases}
\end{aligned}
$$

□

Then for the no-intercept model on $[-1, 1]$ with even degree d and even $d-p$, we have the following theorem.

Theorem 3.2. *Let x_ν , $\nu = 0, 1, ..., d-1$ be the extreme points of the function $u_d(x)$ over $[-1, 1]$. For the no-intercept model on $[-1, 1]$ with even degree d and even $d-p$, the c_p-optimal design is supported on the points $\{x_\nu\}$.*

Proof. We want to find the sign of the determinant $D_\nu(c_p)$, and denote it by $d_\nu(c_p)$, $\nu = 0, 1, ..., d-1$.

$$
\begin{aligned}
D_\nu(c_p) &= |f(x_0) \cdots f(x_{\nu-1}) f(x_{\nu+1}) \cdots f(x_{d-1}) c_p| \\
&= (-1)^{p+d} |A^{p,\nu+1}| \prod_{\substack{t=0 \\ t \neq \nu}}^{d-1} x_t \quad ,
\end{aligned}
$$

where

$$
A^{p,\nu+1} = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_0 & \cdots & x_{\nu-1} & x_{\nu+1} & \cdots & x_{d-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_0^{p-2} & \cdots & x_{\nu-1}^{p-2} & x_{\nu+1}^{p-2} & \cdots & x_{d-1}^{p-2} \\ x_0^{p} & \cdots & x_{\nu-1}^{p} & x_{\nu+1}^{p} & \cdots & x_{d-1}^{p} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_0^{d-1} & \cdots & x_{\nu-1}^{d-1} & x_{\nu+1}^{d-1} & \cdots & x_{d-1}^{d-1} \end{pmatrix} .
$$

$A^{p,\nu+1}$ is the $(d-1) \times (d-1)$ matrix and it's the same by crossing out the p^{th} row and

$(\nu+1)^{th}$th column of the Vandermonde matrix X in (1). The determinant of the minor matrix $A^{p,\nu+1}$ is denoted by $|A^{p,\nu+1}| = |X_{p,(\nu+1)}|$. Since X is a $d \times d$ matrix with

$$|X| = \prod_{0 \le t_1 < t_2 \le d-1} (x_{t_2} - x_{t_1}) \quad ,$$

where $\{x_\nu\}$ are distinct, then $det(X) \neq 0$. Therefore X is invertible and by (4)

$$b_{\nu+1,p} = \frac{a_{\nu,p-1}}{P_\nu(x_\nu)} \quad .$$

On the other hand, by the usual formula in computing the inverse. we have

$$b_{\nu+1,p} = \frac{1}{|X|}[(-1)^{p+\nu+1}|X_{p,\nu+1}|]' \quad ,$$

which in turns implies

$$|A^{p,\nu+1}| = (-1)^{p+\nu+1}\frac{a_{\nu,p-1}}{P_\nu(x_\nu)} \prod_{0 \le t_1 < t_2 \le d-1} (x_{t_2} - x_{t_1}) \quad .$$

when d and p are even, we get

$$D_\nu(c_p) = (-1)^{p+\nu+1}\frac{a_{\nu,p-1}}{P_\nu(x_\nu)} \prod_{\substack{t=0 \\ t \neq \nu}}^{d-1} x_t \prod_{0 \le t_1 < t_2 \le d-1} (x_{t_2} - x_{t_1}) \quad . \tag{5}$$

Now we need to find the sign of $a_{\nu,p-1}$. As we know $a_{\nu,p-1}$ is the coefficient of x^{p-1} in $P_\nu(x)$ in (2). Recall that the extreme points of the $u_d(x)$ are symmetric around 0 and denote them as

$$-x_\nu = x_{d-1-\nu} \quad , \quad \nu = 0, 1, \cdots, \frac{d}{2} - 1 \quad .$$

Expect for $(x - x_{d-\nu}) = (x + x_\nu)$, the factors in $P_\nu(x)$ come in pairs of $x - x_k$ and $x + x_k$. Hence we have

$$\begin{aligned} P_\nu(x) &= (x-x_0)(x-x_1)\cdots(x-x_{\nu-1})(x-x_{\nu+1})\cdots(x-x_{d-2})(x-x_{d-1}) \\ &= (x+x_\nu) \prod_{k=\{0,1,\cdots,\frac{d}{2}-1\}\setminus\{\nu,d-1-\nu\}} (x^2 - x_k^2) \quad . \end{aligned}$$

Multiplying out the products, we find the sign of the coefficient of the odd power x^{p-1} in $P_\nu(x)$ is the same as the sign of

$$\sum_{\substack{\tau \subseteq \{0,1,\cdots,\frac{d}{2}-1\}\backslash\{\nu, d-1-\nu\} \\ \sharp\tau = (\frac{d}{2}-\frac{p}{2})}} \prod_{k\in\tau}(-x_k^2)$$

and it equals to

$$(-1)^{\frac{d}{2}-\frac{p}{2}} \sum_{\tau} \prod_{k\in\tau} x_k^2 \quad . \tag{6}$$

Next it is known that

$$P_\nu(x_\nu) = \prod_{\substack{k=0 \\ k\neq\nu}}^{d-1} (x_\nu - x_k) = (-1)^{d-\nu-1} \prod_{\substack{k=0 \\ k\neq\nu}}^{d-1} |x_\nu - x_k| \tag{7}$$

and

$$sgn(\prod_{\substack{t=0 \\ t\neq\nu}} x_t) = \begin{cases} (-1)^{\frac{d}{2}-1} & , \quad 0 \le \nu \le \frac{d}{2}-1 \\ (-1)^{\frac{d}{2}} & , \quad \frac{d}{2}-1 < \nu \le d-1 \quad . \end{cases} \tag{8}$$

From (6)-(8), the sign of $D_\nu(c)$ in (5) can be established, then we have the following result

$$d_\nu(c_p) = \begin{cases} (-1)^{\frac{p}{2}-1} & , \quad 0 \le \nu \le \frac{d}{2}-1 \\ (-1)^{\frac{p}{2}} & , \quad \frac{d}{2}-1 < \nu \le d-1 \quad . \end{cases}$$

By Studden (1968), we know that

$$\beta c_p = \sum_{\nu=0}^{d-1} (-1)^{d-1-\nu} d_\nu(c_p) p_\nu f(x_\nu) = \sum_{\nu=0}^{d-1} \epsilon_\nu p_\nu f(x_\nu)$$

where β is a positive constant, and

$$\epsilon_\nu = (-1)^{d-1-\nu} d_\nu(c_p) = \begin{cases} (-1)^{d-\nu+\frac{p}{2}} & , \quad 0 \le \nu \le \frac{d}{2}-1 \\ (-1)^{d-\nu+\frac{p}{2}-1} & , \quad \frac{d}{2}-1 < \nu \le d-1 \quad . \end{cases}$$

Write $u_d(x) = \sum_{\nu=0}^{d-1} a_\nu f_\nu(x)$, with coefficient vector $a = (a_0, a_1, ..., a_{d-1})$, $|u_d(x)| \le 1$, $x \in [-1,1]$, and by Lemma 3.2

$$u_d(x_\nu) = \begin{cases} (-1)^{\nu+\frac{p}{2}} & , \qquad 0 \le \nu \le \frac{d}{2} - 1 \\ (-1)^{\nu+\frac{p}{2}-1} & , \qquad \frac{d}{2} - 1 < \nu \le d - 1 \end{cases} .$$

Then

$$\epsilon_\nu u_d(x_\nu) = (-1)^{d+p} = 1 ,$$

and $p_\nu \geq 0$, $\sum_{\nu=0}^{d} p_\nu = 1$, such that

$$\begin{aligned} a\beta c_p &= \sum_{\nu=0}^{d-1} (-1)^{d-1-\nu} d_\nu(c_p) p_\nu (af(x_\nu)) \\ &= \sum_{\nu=0}^{d-1} \epsilon_\nu p_\nu u_d(x_\nu) \\ &= \sum_{\nu=0}^{d-1} p_\nu = 1 . \end{aligned}$$

Hence βc_p is a boundary point of R. Then by Elfving Theorem(1952), the design is c_p-optimal design. $\Box$

Now we consider the case that $d - p$ is odd, we also divide the discussions into two parts according to degree d is even or odd.

For even degree d, Huang and Chen (1996) found an equioscillating function of one lower order and all extrema are symmetric around 0. They also found that the number of the extrema is equal to the number of the unknown parameters, this is presented in Theorem 3.3 for completeness.

Theorem 3.3. *For the no-intercept model of even degree d on $[-1, 1]$ and $d-p$ odd, there exists an equioscillating function $e(x)$ defined in terms of $\{|x|, |x|x, \cdots, |x|x^{d-2}\}$ on $[-1, 1]$ such that the extreme points of the $e(x)$ are $\rho_\nu = cos(\frac{\nu\pi}{d-1})$, $\nu = 0, 1, \cdots, d-1$. Then a c_p-optimal design is supported on the Chebyshev points $\rho_\nu = cos(\frac{\nu\pi}{d-1})$, $\nu = 0, 1, \cdots, d-1$.*

For the case odd d and $d - p$, we have the following theorem.

Theorem 3.4. *For the no-intercept model on $[-1,1]$ with odd degree d and odd $d-p$, the c_p-optimal design are supported on at the $d-1$ alternating extreme points of the function of one lower order $u_{d-1}(x)$.*

Proof. Let $c_p = (\hat{c_p}', 0)'$, and $d' = d-1$ is even. From the proof of Theorem 3.2, we know that a $\hat{c_p}$-optimal design $\hat{\xi}$ is a symmetric design with the support points $x_0, x_1, ..., x_{d'-1}$ which are the extrema of $u_{d'}(x)$ over $[-1,1]$. Let $u_{d'}(x) = \sum_{\nu=0}^{d-1} a_\nu f_\nu(x)$, with coefficient vector $a = (a_0, a_1, ..., a_{d-2}, 0)$, and $f(x) = (f_0(x), f_1(x), ..., f_{d-1}(x))'$, where

$$u_{d'}(x_\nu) = \begin{cases} (-1)^{\nu+\frac{p}{2}} & , \quad 0 \le \nu \le \frac{d'}{2} - 1 \\ (-1)^{\nu+\frac{p}{2}-1} & , \quad \frac{d'}{2} - 1 < \nu \le d' - 1 \ . \end{cases}$$

Then there exists a $\beta > 0$, such that

$$\beta \hat{c_p} = \sum_{\nu=0}^{d'-1} \epsilon_\nu p_\nu (f_0(x_\nu), f_1(x_\nu), ..., f_{d'-1}(x_\nu))' \ ,$$

since $\sum_{\nu=0}^{d'-1} \epsilon_\nu p_\nu f_{d-1}(x_\nu) = 0$, We have that

$$\beta c_p = \sum_{\nu=0}^{d'-1} \epsilon_\nu p_\nu f(x_\nu) \ .$$

And

$$\epsilon_\nu = (-1)^{d'-1-\nu} d_\nu(\hat{c_p}) = \begin{cases} (-1)^{d'-\nu+\frac{p}{2}} & , \quad 0 \le \nu \le \frac{d'}{2} - 1 \\ (-1)^{d'-\nu+\frac{p}{2}-1} & , \quad \frac{d'}{2} - 1 < \nu \le d' - 1 \ , \end{cases}$$

such that

$$\epsilon_\nu u_{d'}(x_\nu) = (-1)^{d'+p} = 1 \ .$$

From the proof of Theorem 3.2, we know that

$$a\beta c_p = 1 \ .$$

Hence $\beta c_p = \sum_{\nu=0}^{d'-1} (-1)^{d'-1-\nu} d_\nu(c_p) p_\nu f(x_\nu)$, and βc_p is a boundary point of R. Then by Elfving theorem, the theorem is proofed. $\square$

3.2 Individual regression coefficients for the no-intercept model over [-b,b]

A difficulty with the c_p-optimal design for no intercept model is that it is not invariant under linear transformations, that is the optimal design on a given interval does not transform linearly. Although it is of interest to know whether it would be scale invariant. Next we will prove the scale invariance property of c_p-optimality. Some properties of the information matrices between two designs with scale supporting points and the same corresponding weights will be presented below.

Lemma 3.3. *Two designs* $\xi_1 = \begin{Bmatrix} x_1, x_2, \cdots, x_n \\ p_1, p_2, \cdots, p_n \end{Bmatrix}$ *and* $\xi_2 = \begin{Bmatrix} bx_1, bx_2, \cdots, bx_n \\ p_1 \ , \ p_2 \ , \cdots, \ p_n \end{Bmatrix}$, $b \in R$, $-1 \le x_1 < x_2 < \cdots < x_n \le 1$, *with information matrices* $M(\xi_1)$ *and* $M(\xi_2)$, *and* $f(x) = (x, \cdots, x^d)'$. *Then the determinant*

$$|M(\xi_2)| = b^{d(d+1)}|M(\xi_1)| \quad ,$$

and the inverse $M^{-1}(\xi_1) = [a_{ij}]$, $M^{-1}(\xi_2) = [e_{ij}]$ *such that* $[e_{ij}] = b^{-(i+j)}[a_{ij}]$.

Proof. Let $c_i = \int_{-1}^{1} x^i d\xi_1(x)$, and

$$M(\xi_1) = \begin{pmatrix} c_2 & c_3 & \cdots & c_{d+1} \\ \vdots & \vdots & & \vdots \\ c_{d+1} & c_{d+2} & \cdots & c_{2d} \end{pmatrix}$$

then

$$\begin{aligned} |M(\xi_2)| &= det \begin{pmatrix} b^2c_2 & b^3c_3 & \cdots & b^{d+1}c_{d+1} \\ \vdots & \vdots & & \vdots \\ b^{d+1}c_{d+1} & b^{d+2}c_{d+2} & \cdots & b^{2d}c_{2d} \end{pmatrix} \\ &= b^{\frac{d(d+3)}{2}} det \begin{pmatrix} c_2 & bc_3 & \cdots & b^{d-1}c_{d+1} \\ \vdots & \vdots & & \vdots \\ c_{d+1} & bc_{d+2} & \cdots & b^{d-1}c_{2d} \end{pmatrix} \\ &= b^{\frac{d(d+3)}{2}} b^{\frac{d(d-1)}{2}} |M(\xi_1)| \\ &= b^{d(d+1)}|M(\xi_1)| \end{aligned}$$

Now we need to find the inverse, let $|M(\xi_2)_{ij}|$ denote the determinant of the minor matrix of $M(\xi_2)$ and $adj(M(\xi_2))$ be the transposed matrix of cofactors. Then

$$\begin{aligned}
M^{-1}(\xi_2) &= \frac{1}{|M(\xi_2)|} adj(M(\xi_2)) \\
&= \frac{1}{b^{d(d+1)}|M(\xi_1)|}[(-1)^{i+j}|M(\xi_2)_{ij}|]' \\
&= \frac{1}{b^{d(d+1)}|M(\xi_1)|}\ [(-1)^{i+j} b^{d(d+1)-(i+j)}|M(\xi_1)_{ij}|]' \\
&= \frac{1}{|M(\xi_1)|}[b^{-(i+j)}(-1)^{i+j}|M(\xi_1)_{ij}|]' \quad .
\end{aligned}$$

□

Now we can prove the scale invariance of c_p-optimal with Lemma 3.3.

Theorem 3.5. *For the no-intercept model, if the design ξ_1 is c_p-optimal over $[-1,1]$ with support points $\{x_\nu\}$ and corresponding weights $\{p_\nu\}$. Then ξ_2 is c_p-optimal design over $[-b,b]$ with support points $\{bx_\nu\}$ and corresponding weights $\{p_\nu\}$, $b > 0$.*

Proof. Since the design ξ_1 is c_p-optimal, such that for all $x \in [-1,1]$,

$$(c_p' M^{-1}(\xi_1) f(x))^2 \le c_p' M^{-1}(\xi_1) c_p$$

Let $y = bx$, $y \in [-b,b]$, $b > 0$,

$$\begin{aligned}
c_p' M^{-1}(\xi_2) f(y) &= (0,\cdots,0,1,0,\cdots,0)\begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1d} \\ \vdots & \vdots & & \vdots \\ e_{d1} & e_{d2} & \cdots & e_{dd}\end{pmatrix}\begin{pmatrix} y \\ \vdots \\ y^d\end{pmatrix} \\
&= (e_{p1},\cdots,e_{pd})\begin{pmatrix} y \\ \vdots \\ y^d\end{pmatrix} \\
&= \sum_{j=1}^{d} e_{pj} y^j = \sum_{j=1}^{d} b^{-(p+j)} a_{pj} y^j \\
&= b^{-p}\sum_{j=1}^{d} a_{pj}(\frac{y}{b})^j = b^{-p}\sum_{j=1}^{d} a_{pj} x^j
\end{aligned}$$

$$= b^{-p}c_p'M^{-1}(\xi_1)f(x)$$

and

$$\begin{aligned} c_p'M^{-1}(\xi_2)c_p &= e_{pp} = b^{-2p}a_{pp} \\ &= b^{-2p}c_p'M^{-1}(\xi_1)c_p \end{aligned}$$

Then

$$\begin{aligned} (c_p'M^{-1}(\xi_2)f(y))^2 &= b^{-2p}(c_p'M^{-1}(\xi_1)f(x))^2 \\ &\leq b^{-2p}(c_p'M^{-1}(\xi_1)c_p) \\ &= b^{-2p}b^{2p}(c_p'M^{-1}(\xi_2)c_p) \\ &= c_p'M^{-1}(\xi_2)c_p \end{aligned}$$

By the Equivalence Theorem, it is shown that ξ_2 is c_p-optimal. $\square$

3.3 Individual regression coefficients for the no-intercept model over [a,b]

In this section we will investigate the c_p-optimal designs for no intercept model on an arbitrary compact interval $[a, b]$. By fixing $b = 1$, we discuss the problem of finding the c_p-optimal design for no intercept model on the interval $[a, 1],\ -1 \leq a < 1$. The c_p-optimal design for the model $f(x) = (x, x^2, \cdots, x^d)$ on $[a, 1]$ for estimating p^{th} parameter is denoted by $\xi^p_{a,d}$.

For 2^{nd} degree polynomial regression model over nonsymmetric interval, we find the c_p-optimal design by Elfving set (Figure 2). We only list the c_p-optimal designs over the interval $[a, 1]$.

For the case $d = 2$ and $p = 1$ over $[a, 1]$, we know that the regression polynomial $f(x) = (x, x^2)'$ and $c_p = (1, 0)'$. We would like to estimate the coefficient of x and divide the different intervals into 3 parts according to the value of a, where the optimal design $\xi^1_{a,2}$ has the following patterns.

Theorem 3.6. *For the model $f(x) = (x, x^2)'$ defined on $[a, 1]$ estimating 1^{st} parameter, the design $\xi^1_{a,2}$ supported on the two points a and 1 is c_p-optimal if $-1 \le a \le 1 - \sqrt{2}$ or $-1 + \sqrt{2} \le a < 1$, with corresponding weights $1/(1 + a^2),\ a^2/(1 + a^2)$.*

Moreover, if $1 - \sqrt{2} \le a \le 2\sqrt{2} - 3$, the c_p-optimal design $\xi^1_{a,2}$ supported on the two points $a,\ -(1 + \sqrt{2})a$ with corresponding weights $(2 + \sqrt{2})/4,\ (2 - \sqrt{2})/4$.

If $2\sqrt{2} - 3 < a \le -1 + \sqrt{2}$, the c_p-optimal design $\xi^1_{a,2}$ supported on the two fixed points $-1 + \sqrt{2},\ 1$ with corresponding weights $(2 + \sqrt{2})/4,\ (2 - \sqrt{2})/4$.

Proof. At Figure 2 (a)-(d), we give the Elfving sets for 2^{nd} degree polynomial model without intercept over $[a, 1]$, $-1 \le a \le 1 - \sqrt{2}$. The equation $y = 2(x - 1) + 1$ is the tangent line of $y = x^2$ with tacnode $(1, 1)$, and the intersection of the equation and $y = -x^2$ are $(-1 + \sqrt{2}, -3 + 2\sqrt{2})$ and $(-1 - \sqrt{2}, -3 - 2\sqrt{2})$, but $(-1 - \sqrt{2}, -3 - 2\sqrt{2})$ does not belong to the domain. Since the point $(-1 + \sqrt{2}, -3 + 2\sqrt{2})$ is on $y = -x^2$, we find the symmetric point of it to origin is $(1 - \sqrt{2}, 3 - 2\sqrt{2})$. Thus by Elfving Theorem we know that if $-1 \le a \le 1 - \sqrt{2}$, the c_p-optimal is supported on the two endpoints a and 1. Then we need to find the corresponding weights, by Theorem 2.2, we can find that

$$(X')^{-1} = \begin{pmatrix} \dfrac{1}{a - a^2} & \dfrac{-1}{a - a^2} \\ \dfrac{-a^2}{a - a^2} & \dfrac{a}{a - a^2} \end{pmatrix} \text{ and } u = \begin{pmatrix} \dfrac{1}{a - a^2} \\ \dfrac{-a^2}{a - a^2} \end{pmatrix},$$

thus the corresponding weights are $1/(1 + a^2),\ a^2/(1 + a^2)$.

Similarly, if $-1 + \sqrt{2} \le a < 1$, the c_p-optimal is supported on the two endpoints a and 1 with corresponding weights $1/(1 + a^2),\ a^2/(1 + a^2)$.

For the interval $a \ge 1 - \sqrt{2}$, the tangent line could not pass $(1, 1)$ any more. Let (b, b^2)

be the tacnode of the tangent line on $y = x^2$ passing $(-a, -a^2)$, then we can find the slope of the tangent line is $2b$, which equal to $[b^2 - (-a^2)]/[b - (-a)]$, thus

$$b = -(1 + \sqrt{2})a. \tag{9}$$

Then we need to find at which value of a, the tangent line would pass the point $(-1, -1)$. Thus the slope of the tangent line is equal to $[b^2 - (-1)]/[b - (-1)]$, then $b = -1 \pm \sqrt{2}$, but $b = -1 - \sqrt{2}$ does not belong to the domain. From (9) we know that $a = 2\sqrt{2} - 3$. By Elfving Theorem we know that if $1 - \sqrt{2} \le a \le 2\sqrt{2} - 3$, the c_p-optimal design $\xi^1_{a,2}$ supported on $a,\ -(1 + \sqrt{2})a$.

By Theorem 2.2 we can find that

$$(X')^{-1} = \begin{pmatrix} \dfrac{(1+\sqrt{2})^2}{(4+3\sqrt{2})a} & \dfrac{(1+\sqrt{2})}{(4+3\sqrt{2})a^2} \\ \dfrac{-1}{(4+3\sqrt{2})a} & \dfrac{1}{(4+3\sqrt{2})a^2} \end{pmatrix} \text{ and } u = \begin{pmatrix} \dfrac{(1+\sqrt{2})^2}{(4+3\sqrt{2})a} \\ \dfrac{-1}{(4+3\sqrt{2})a} \end{pmatrix},$$

thus the corresponding weights are $(2 + \sqrt{2})/4,\ (2 - \sqrt{2})/4$.

For the interval $a > 2\sqrt{2} - 3$, the tangent line keep passing $(-1, -1)$ with tacnode (b, b^2) until $a = -1 + \sqrt{2}$, where $b = -1 + \sqrt{2}$. By Elfving Theorem we know that if $2\sqrt{2} - 3 < a \le -1 + \sqrt{2}$, the c_p-optimal design $\xi^1_{a,2}$ supported on the two fixed points $-1 + \sqrt{2},\ 1$. Thus we need to find the corresponding weights, we can use the result $1/(1 + a^2)$, wheree $a = -1 + \sqrt{2}$, such that the corresponding weights are $(2 + \sqrt{2})/4,\ (2 - \sqrt{2})/4$. $\square$

For the case $d = 2$ and $p = 2$ over $[a, 1]$, that is $f(x) = (x, x^2)'$ and $c_p = (0, 1)'$.We would like to estimate the coefficient of x^2 and divide the different intervals into 2 parts according to the value of a.

Theorem 3.7. *For the model $f(x) = (x, x^2)'$ defined on $[a, 1]$ estimating 2^{nd} parameter, the design $\xi^2_{a,2}$ supported on the two points a and 1 is c_p-optimal if $-1 \le a \le 2\sqrt{2} - 3$ or*

$-1+\sqrt{2} \le a < 1$, *with corresponding weights* $w_1, 1-w_1$, *where*

$$w_1 = \begin{cases} \dfrac{1}{1-a} & , \quad if \quad -1 \le a \le 2\sqrt{2}-3 \\ \dfrac{1}{1+a} & , \quad if \quad -1+\sqrt{2} \le a < 1 \end{cases} .$$

Moreover, if $2\sqrt{2}-3 \le a \le -1+\sqrt{2}$, *the* c_p*-optimal design* $\xi^2_{a,2}$ *supported on the two points* $-1+\sqrt{2},\ 1$ *with corresponding weights* $\sqrt{2}/2, (2-\sqrt{2})/2$.

Proof. By Theorem 3.6 we know that the tangent line on $y = -x^2$ passing $(1,1)$ have intersection with $y = x^2$ at $x = 2\sqrt{2}-3$. By Elfving Theorem we know that if $-1 \le a \le 2\sqrt{2}-3$, the c_p-optimal design $\xi^2_{a,2}$ supported on the two endpoints a and 1.

By Theorem 2.2 we can find that

$$(X')^{-1} = \begin{pmatrix} \dfrac{1}{a-a^2} & \dfrac{-1}{a-a^2} \\ \dfrac{-a^2}{a-a^2} & \dfrac{a}{a-a^2} \end{pmatrix} \text{ and } u = \begin{pmatrix} \dfrac{-1}{a-a^2} \\ \dfrac{a}{a-a^2} \end{pmatrix},$$

thus the corresponding weight of a is $1/(1-a)$.

By Theorem 3.6 we know that the tangent line on $y = x^2$ passing $(-1,-1)$ with $(-1+\sqrt{2}, 3-2\sqrt{2})$ as its tacnode. Thus if $-1+\sqrt{2} \le a < 1$, the c_p-optimal $\xi^2_{a,2}$ is supported on the two endpoints a and 1 with corresponding weight of a is $w_1 = 1/(1+a)$.

For the interval $a \ge 2\sqrt{2}-3$, the tangent line keep passing $(1,1)$ with tacnode $(b, -b^2)$ on $y = -x^2$ until $a = -1+\sqrt{2}$, where $b = 1-\sqrt{2}$. By Elfving Theorem we know that if $2\sqrt{2}-3 \le a \le -1+\sqrt{2}$, the c_p-optimal design $\xi^1_{a,2}$ supported on the two fixed points $-1+\sqrt{2},\ 1$. Thus we can find the corresponding weights $\sqrt{2}/2, (2-\sqrt{2})/2$. $\square$

4 Discussion

For the no-intercept model over $[-1, 1]$, Huang and Chen (1996) found c_p-optimal designs as d plus $d-p$ is odd. We found c_p-optimal designs as d plus $d-p$ is even over $[-1, 1]$. It is also shown that the support points are scale invariant over $[-b, b]$. In order to obtain the optimal designs, we either find the sign pattern of the determinants $D_\nu(c_p)$ for each $\nu = 0, 1, \cdots, k$, or show that the conditions of Elfving theorem hold for the vector c_p directly. For the no-intercept model $f(x) = (x, x^2)'$ over nonsymmetric interval $[a, b]$, by fixing $b = 1$, we have obtained c_p-optimal designs when $-1 \leq a < 1$. In the future, it is of interest to find general analytical formulas for nonsymmetric interval.

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Appendix

d		positive support points (symmetric around 0)				
2		1.0				
4		0.643594	1.0			
6		0.442891	0.855600	1.0		
8		0.335671	0.692977	0.920738	1.0	
10		0.269817	0.573649	0.803643	0.949796	1.0
d	p	corresponding weights				
2	2	0.5				
4	2	0.426777	0.073223			
4	4	0.353553	0.146447			
6	2	0.386895	0.083333	0.029772		
6	4	0.283226	0.157229	0.059544		
6	6	0.227671	0.183013	0.089316		
8	2	0.374738	0.070436	0.038582	0.016243	
8	4	0.263181	0.128310	0.076023	0.032486	
8	6	0.204690	0.138086	0.108495	0.048730	
8	8	0.168894	0.135299	0.130834	0.064973	
10	2	0.369382	0.065410	0.032170	0.022787	0.0102509
10	4	0.254561	0.117094	0.062525	0.045319	0.0205017
10	6	0.194949	0.121724	0.085759	0.066816	0.0307526
10	8	0.158717	0.115725	0.098203	0.086353	0.0410034
10	10	0.134458	0.107623	0.103884	0.102781	0.0512543

Table 1: The support points and corresponding weighs for c_p-optimal designs without intercept for even degree d, $d - p$ on $[-1, 1]$.

Table 2: The sign pattern of $D_\nu(c_p)$ for even degree d, $d-p$.

		$d_\nu(c_p)$									
d	p	1	2	3	4	5	6	7	8	9	10
2	2	+	-								
4	2	+	+	-	-						
4	4	-	-	+	+						
6	2	+	+	+	-	-	-				
6	4	-	-	-	+	+	+				
6	6	+	+	+	-	-	-				
8	2	+	+	+	+	-	-	-	-		
8	4	-	-	-	-	+	+	+	+		
8	6	+	+	+	+	-	-	-	-		
8	8	-	-	-	-	+	+	+	+		
10	2	+	+	+	+	+	-	-	-	-	-
10	4	-	-	-	-	-	+	+	+	+	+
10	6	+	+	+	+	+	-	-	-	-	-
10	8	-	-	-	-	-	+	+	+	+	+
10	10	+	+	+	+	+	-	-	-	-	-

Table 3: The value of $u_d(x_\nu)$ for even degree d, $d-p$.

		$u_d(x_\nu)$									
d	p	1	2	3	4	5	6	7	8	9	10
2	2	+	-								
4	2	-	+	+	-						
4	4	+	-	-	+						
6	2	+	-	+	+	-	+				
6	4	-	+	-	-	+	-				
6	6	+	-	+	+	-	+				
8	2	-	+	-	+	+	-	+	-		
8	4	+	-	+	-	-	+	-	+		
8	6	-	+	-	+	+	-	+	-		
8	8	+	-	+	-	-	+	-	+		
10	2	+	-	+	-	+	+	-	+	-	+
10	4	-	+	-	+	-	-	+	-	+	-
10	6	+	-	+	-	+	+	-	+	-	+
10	8	-	+	-	+	-	-	+	-	+	-
10	10	+	-	+	-	+	+	-	+	-	+

Figure 1: The plots of $\{c'M^{-1}(\xi)f(x)\}^2$ for $f(x) = (x, \cdots, x^6)$ where estimating p^{th} parameter.

(a) $p = 1$

(b) $p = 2$

(c) $p = 3$

(d) $p = 4$

(e) $p = 5$

(f) $p = 6$

Figure 2: Elfving sets for 2^{nd} degree polynomial model without intercept over $[a, 1]$.

(a) $a = -1$

(d) $a = 1 - \sqrt{2}$

(b) $a = -0.7$

(e) $a = -0.3$

(c) $a = -0.5$

(f) $a = -0.2$

(g) $a = 2\sqrt{2} - 3$

(h) $a = 0$

(i) $a = 0.1$

(j) $a = 0.2$

(k) $a = -1 + \sqrt{2}$

(l) $a = 0.7$