Remarks on a New Inverse Nodal Problem

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In a recent paper, X. F. Yang proved a uniqueness theorem on inverse nodal problems that links to inverse spectral theory, on one hand, and reduces the redundancy of the classical inverse nodal problems, on the other hand. In this note we improve Yang’s theorem by weakening its conditions and simplifying its proof. We also discuss variants of Yang’s theorem.

1. INTRODUCTION

Consider the Sturm–Liouville problem

\[-y'' + q(x)y = \lambda y,\]

subject to boundary conditions

\[y(0)\cos \alpha + y'(0)\sin \alpha = 0,\]
\[y(1)\cos \beta + y'(1)\sin \beta = 0,\]

where \(q \in L^1(0,1)\) and \(\alpha, \beta \in [0, \pi]\). Let \(\lambda_n\) be the \(n\)th eigenvalue of the operator, and let \(0 < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < 1\) be the nodal points of the \(n\)th eigenfunction. Also let \(s_n = \sqrt{\lambda_n}\), and let \(l_k^{(n)} = x_{k+1}^{(n)} - x_k^{(n)}\) be the associated nodal length.

The inverse nodal problem, first studied by McLaughlin, is the problem of finding the potential \(q\) and boundary conditions \(\alpha, \beta\) using only the set of nodal points \(\{x_i^{(n)}\}\). It seems to be a good inverse problem, for not only is \((q, \alpha, \beta)\) uniquely determined by the nodal set up to a constant [2, 5, 11, 14], it can also be reconstructed from the nodal set [5, 8, 14]. Furthermore,
although the inverse nodal problem is overdetermined, it turns out to be well-posed after a partition \([9]\).

The development of the inverse spectral theory, its close relative, is quite different. The set of eigenvalues, in general, is not sufficient to determine the operator, represented by \((q, \alpha, \beta)\). One needs more data to determine \((q, \alpha, \beta)\), for example, two sets of eigenvalues \([1]\), one set of eigenvalues plus a symmetric potential function \([6]\), and one set of eigenvalues plus knowledge of \(q\) on \((0, \frac{1}{2})\) \([7]\). The interested reader may read \([10]\) for a survey. Recently Gesztesy and Simon also produced a number of new results \([3, 4, 12]\).

Using one of Gesztesy and Simon’s results, Yang proved an interesting theorem \([15]\) on inverse nodal problems which, on one hand, makes use of inverse spectral theory and, on the other hand, reduces the redundancy of nodal data. In short, he showed that the set of all nodal points in the subinterval \((0, b)\) \((1/2 < b \leq 1)\) is sufficient to obtain uniqueness.

The purpose of this note is to improve Yang’s theorem \([15, \text{Theorem 2.1}]\), simplify its proof, and also discuss various implications of the theorem.

Fix \(0 < b \leq 1\) and, associated with the Sturm–Liouville problem defined by \((q, \alpha, \beta)\), let \(S = \{n_k\}\) be a strictly increasing sequence of positive integers. Suppose

\[
T(S) = \{(j, n) : n \in S, j = 1, \ldots, n - 1\}.
\]

Let \(A \subset T(S)\) be such that for any \(n_k \in S\), there is some \(j = j_k\) such that \((j_k, n_k) \in A\). Note that there might be only one choice of \(j_k\). Define \(B(A)\) to be a subnodal set on \((0, b)\) if

\[
B(A) = \{x_j^{(n)} \in (0, b) : (j, n) \in A\}.
\]

Clearly \(B(A)\) depends on the problem \((q, \alpha, \beta)\), the interval \((0, b)\), as well as \(S\) and \(A\).

**Definition.** We say that

(a) \(B(A)\) is twin on \((0, b)\) if for any \(k \geq 1\), there is a pair of adjacent nodal points \(x_{j_k}^{(n_k)}\) and \(x_{j_k+1}^{(n_k)}\) contained in \(B(A)\).

(b) \(B(A)\) is dense on \((0, b)\) if \(\text{cl}(B(A)) = [0, b]\).

(c) \(B(A)\) is \(S\)-dense on \((0, b)\) if there is a subsequence of \(S\) (also denoted by \(\{n_k\}\)) such that for any \(x \in (0, b)\) there exists \(\{i_k\}\) such that \(x_{j_k}^{(n_k)} \in B(A)\) such that \(\lim_{k \to \infty} x_{j_k}^{(n_k)} = x\).

Obviously, if \(B(A)\) is \(S\)-dense on \((0, b)\), then it is dense.
Throughout this paper, \( j_k \) is reserved for the first index of the twin nodal points for the \( n_k \)th eigenfunction, in the case of twin subnodal sets. We shall also let \( \{ \lambda_n \} \) and \( \{ x^{(n)}_j \} \) stand for the spectrum and nodal set for the Sturm–Liouville problem defined by \( (\bar{q}, \bar{\alpha}, \bar{\beta}) \).

**Theorem 1.1.** Take \( 0 < b \leq 1 \). Assume that \( B(A) \) is twin and dense on \((0, b)\) about \((q, \lambda, \beta)\). If for any \( k \geq 1 \), there is \((j_k, \bar{n}_k)\) such that

\[
J_{j_k + j}^{(n_k)} = \bar{J}_{j_k + j}^{(n_k)},
\]

for all \( j \in \mathbb{Z} \) such that \( x^{(n_k)}_{j_k + j} \in B(A) \), then except finitely many \( k \)'s,

\[
(j_k, n_k) = (\bar{j}_k, \bar{n}_k), \quad \lambda_{n_k} = \bar{\lambda}_{n_k} + C.
\]

Also,

\[
\alpha = \bar{\alpha}, \quad \text{and} \quad q = \bar{q} + C \text{ a.e. on } (0, b).
\]

Here \( C \) is a real constant.

Note that in [15, Lemma 2.1] which is a key lemma for Yang’s theorem, \( B(A) \) was required to be \( S \)-dense instead of dense. We also note that \( b \) can be arbitrarily small. Our proof will be simpler and more direct. It will be given in Section 2. In Section 3, we shall discuss the applications of this theorem.

2. PROOF OF MAIN THEOREM

We first quote some lemmas for the asymptotic estimates of \( s_n = \sqrt{\lambda_n} \), the nodal points \( \{ x^{(n)}_j \} \), and nodal lengths \( \{ l^{(n)}_j \} \) corresponding to \( (q, \alpha, \beta) \). The following lemma is reproduced from [8, Lemma 2.1; 14, Lemma 2.8]. Note that there are some typo errors in [8].

**Lemma 2.1.** Suppose that \( (q, \alpha, \beta) \in L^1(0, 1) \times [0, \pi)^2 \). Let \( \alpha_0 = 1 \) if \( \alpha > 0 \), and \( \alpha_0 = -1 \) if \( \alpha = 0 \).

(a) If \( \alpha = \beta = 0 \) or \( \alpha, \beta > 0 \), then as \( n \to \infty \)

\[
s_n = n \pi + \frac{1}{n \pi} \left( -\cot \alpha + \cot \beta + \frac{1}{2} \int_0^1 (1 + \alpha_0 \cos(2n \pi t))q(t) \, dt \right) + O\left( \frac{1}{n^2} \right).
\]
(b) If $\alpha = 0 < \beta$ or $\alpha > 0 = \beta$, then as $n \to \infty$

$$s_n = \left(n - \frac{1}{2}\right)\pi + \frac{1}{(n - \frac{1}{2})\pi}\left(-\cot \alpha + \cot \beta\right)$$

$$+ \frac{1}{2} \int_{0}^{1} (1 + \alpha_0 \cos((2n - 1)\pi t))q(t) \, dt + O\left(\frac{1}{n^2}\right).$$

Here $\cot \gamma = 0$ if $\gamma = 0$; $= \cot \gamma$ otherwise.

**Lemma 2.2.** Suppose $(q, \alpha, \beta) \in L^1(0, 1) \times [0, \pi]^2$. Then as $n \to \infty$, with $1 < j < n - 1$,

(a)

$$I_j^{(n)} = \begin{cases} \frac{1}{n} + o\left(\frac{1}{n^2}\right), & \text{if (I) } \alpha = \beta = 0 \text{ or } \alpha \beta > 0 \\ \frac{1}{n - \frac{1}{2}} + o\left(\frac{1}{n^2}\right), & \text{if (II) } \alpha = 0 < \beta \text{ or } \beta = 0 < \alpha. \end{cases}$$ (2.1)

(b)

$$x_j^{(n)} = \begin{cases} \frac{j}{n} + O\left(\frac{1}{n^2}\right), & \text{if (Ia) } \alpha = \beta = 0 \\ \frac{j - \frac{1}{2}}{n} + O\left(\frac{1}{n^2}\right), & \text{if (Ib) } \alpha \beta > 0 \\ \frac{j}{n - \frac{1}{2}} + O\left(\frac{1}{n^2}\right), & \text{if (IIa) } \alpha = 0 < \beta \\ \frac{j - \frac{1}{2}}{n - \frac{1}{2}} + O\left(\frac{1}{n^2}\right), & \text{if (IIb) } \beta = 0 < \alpha. \end{cases}$$ (2.2)

The lemma above can be derived from [8, Lemma 2.2]. Part (a) of Theorem 2.3 is due to [14, Theorem 1.2]; part (b) is due to [8].
THEOREM 2.3. The reconstruction formulas of \((q, \alpha, \beta)\) from the nodal set are given by:

(a) (i) Either \(\alpha = 0\), or with \(i/n\) tending to 0,

\[
\cot \alpha = \begin{cases} 
\lim_{n \to \infty} n\pi^2 \left( i - \frac{1}{2} - nx_{i}^{(n)} \right), & \text{if } \beta \neq 0 \\
\lim_{n \to \infty} \left( n - \frac{i}{2} \right) \pi^2 \left( i - \frac{1}{2} - (n - \frac{1}{2})x_{i}^{(n)} \right), & \text{if } \beta = 0.
\end{cases}
\] (2.3)

(ii) Either \(\beta = 0\), or with \(i/n\) tending to 0,

\[
\cot \beta = \begin{cases} 
\lim_{n \to \infty} n\pi^2 \left( n - \frac{i}{2} - nx_{i-1}^{(n)} \right), & \text{if } \alpha \neq 0 \\
\lim_{n \to \infty} n\pi^2 \left( n - \frac{i}{2} - (n - \frac{1}{2})x_{i-1}^{(n)} \right), & \text{if } \alpha = 0.
\end{cases}
\]

(b) (i) If \((\alpha, \beta) \in \text{case (I) (cf. (2.1))}, then for a.e. \(x \in (0, 1)\), with \(x_{i}^{(n)}\) convergent to \(x\),

\[
q(x) = \lim_{n \to \infty} 2n^2 \pi^2 \left( nl_{j}^{(n)} - 1 + \frac{j^{(n)}}{n\pi^2} \left( \frac{1}{2} \int_{0}^{1} q \cot \alpha + \cot \beta \right) \right).
\] (2.4)

(ii) If \((\alpha, \beta) \in \text{case (II)}, then for a.e. \(x \in (0, 1)\), with \(x_{i}^{(n)}\) convergent to \(x\),

\[
q(x) = \lim_{n \to \infty} 2 \left( n - \frac{1}{2} \right)^2 \pi^2 \\
\times \left( n - \frac{1}{2} \right) l_{j}^{(n)} - 1 + \frac{j^{(n)}}{n\pi^2} \left( \frac{1}{2} \int_{0}^{1} q \cot \alpha + \cot \beta \right) \right).
\] (2.5)

Here \(\cot \gamma = 0 \text{ if } \gamma = 0; = \cot \gamma \text{ otherwise}.

LEMMA 2.4. Let \(0 < b \leq 1\). Suppose \(B(A)\) is twin on \((0, b)\), and for all \(k \geq 1\), there is \((j_{k}, n_{k})\) such that

\[
x_{j_{k}}^{(n_{k})} = \tilde{x}_{j_{k}}^{(n_{k})} \quad \text{and} \quad x_{j_{k} + 1}^{(n_{k})} = \tilde{x}_{j_{k} + 1}^{(n_{k})}.
\]

Then except for finitely many \(k\)'s,

\[
(j_{k}, n_{k}) = (\tilde{j}_{k}, \tilde{n}_{k}).
\]
Moreover, the boundary conditions \((\alpha, \beta)\) and \((\bar{\alpha}, \bar{\beta})\) belong to the same subcase in (2.2).

**Proof.** By the asymptotic expression of \(l_j^{(n)}\) in (2.1), in both cases,
\[
l_j^{(n_k)} = x_j^{(n_{k+1})} - x_j^{(n_k)},
\]
\[
= \frac{1}{n_k} + O\left(\frac{1}{n_k}\right),
\]
\[
= \frac{1}{n_k} + o\left(\frac{1}{n_k}\right).
\]
Similarly,
\[
l_j^{(\bar{n}_k)} = \frac{1}{\bar{n}_k} + o\left(\frac{1}{n_k}\right).
\]
Thus given \(\epsilon > 0\), when \(k\) is sufficiently large,
\[
\frac{1 - \epsilon}{n_k} < l_j^{(n_k)} < \frac{1 + \epsilon}{n_k},
\]
and
\[
\frac{1 - \epsilon}{\bar{n}_k} < l_j^{(\bar{n}_k)} < \frac{1 + \epsilon}{\bar{n}_k}.
\]
Hence
\[
\frac{1 - \epsilon}{1 + \epsilon} < \frac{n_k}{\bar{n}_k} < \frac{1 + \epsilon}{1 - \epsilon}.
\]
This is valid for any \(\epsilon > 0\). So we have
\[
\lim_{k \to \infty} \frac{\bar{n}_k}{n_k} = 1. \tag{2.6}
\]

Next, we show that \((\alpha, \beta)\) and \((\bar{\alpha}, \bar{\beta})\) belong to the same case in (2.1). Suppose not; say, \((\alpha, \beta)\) belongs to case (I) while \((\bar{\alpha}, \bar{\beta})\) belongs to case (II). Then as \(k \to \infty\), by (2.1),
\[
\frac{1}{n_k} + o\left(\frac{1}{n_k}\right) = \frac{1}{\bar{n}_k - \frac{1}{2}} + o\left(\frac{1}{\bar{n}_k^2}\right).
\]
Thus by (2.6),
\[
\frac{\bar{n}_k - n_k - \frac{1}{2}}{n_k (\bar{n}_k - \frac{1}{2})} = o \left( \frac{1}{n_k^2} \right).
\]
This is impossible, since \( n_k \bar{n}_k \approx n_k^2 \) and \( n_k, \bar{n}_k \in \mathbb{N} \). Hence \((\alpha, \beta)\) and \((\bar{\alpha}, \bar{\beta})\) have to belong to the same case, say, case (I). So
\[
\frac{1}{n_k} + o \left( \frac{1}{n_k^2} \right) = \frac{1}{\bar{n}_k} + o \left( \frac{1}{\bar{n}_k^2} \right),
\]
which implies,
\[
n_k - \bar{n}_k = o(1).
\]
Therefore if \( k \) is sufficiently large, then \( n_k = \bar{n}_k \).

Now comparing the asymptotic expression for \( x_k^{(n_k)} \) and \( \bar{x}_k^{(\bar{n}_k)} \) in (2.2), we conclude that \((\alpha, \beta)\) and \((\bar{\alpha}, \bar{\beta})\) belong to the same subcase in (2.2) and \( j_k = \bar{j}_k \) for sufficiently large \( k \).

**Proof of Theorem 1.1.** First we apply Lemma 2.4 and the reconstruction formula (2.3) for \( \cot \alpha \) to see directly that \( \alpha = \bar{\alpha} \).

Then we see that, from the asymptotic expressions of \( s_n = \sqrt{\lambda_n} \) (Lemma 2.1), we have
\[
\lambda_n = \begin{cases} 
\frac{n^2 \pi^2 + 2 \cot \beta - 2 \cot \alpha + \int_0^1 q + o(1)}{2} & \text{if } (\alpha, \beta) \in \text{case (I)} \\
(n - \frac{1}{2}) \frac{\pi^2 + 2 \cot \beta - 2 \cot \alpha + \int_0^1 q + o(1)}{2} & \text{if } (\alpha, \beta) \in \text{case (II)}.
\end{cases}
\]

Now that \( \alpha = \bar{\alpha} \), we have, when \( k \) is large enough,
\[
\lambda_{n_k} - \bar{\lambda}_{n_k} = C + o(1),
\]
where
\[
C = 2(\cot \beta - \cot \bar{\beta}) + \int_0^1 (q - \bar{q}).
\]

Let \( \phi_n \) and \( \bar{\phi}_n \) be the normalized eigenfunctions for the Sturm–Liouville problems defined by \((q, \alpha, \beta)\) and \((\bar{q}, \bar{\alpha}, \bar{\beta})\), respectively, such that
\[
\phi_n(0) = \bar{\phi}_n(0) = \sin \alpha, \quad \phi'_n(0) = \bar{\phi}'_n(0) = -\cos \alpha.
\]
Then, by the Lagrange identity, when \( k \) is large,
\[
\left( \phi_n'(x) \overline{\phi_n}(x) - \phi_n(x) \overline{\phi_n}'(x) \right) = \left( q(x) - \overline{q}(x) + \lambda_n - \lambda_n \right) \phi_n(x) \overline{\phi_n}(x).
\]
(2.7)

Fix \( x \in (0, b) \). Since \( B(A) \) is dense in \( (0, b) \), then either there is a sequence of nodal points \( x^{(n_k)}_i \in B(A) \) convergent to \( x \) or \( x \in B(A) \). Hence we may integrate (2.7) from 0 to \( x^{(n_k)}_i \) (or \( x \) for the later case) to obtain
\[
0 = \int_0^{x^{(n_k)}_i} \left( \phi_n'(t) \overline{\phi_n}(t) - \phi_n(t) \overline{\phi_n}'(t) \right) dt,
\]
\[
= \int_0^{x^{(n_k)}_i} \left( q(t) - \overline{q}(t) + C + o(1) \right) \phi_n(t) \overline{\phi_n}(t) dt.
\]
(2.8)

It is known that, from [13, Lemma 1.7] and \( 1/s_{n_k} - 1/s_{n_k} = O(1/n_k) \),
\[
\phi_n(x) \overline{\phi_n}(x) = \begin{cases} 
\cos(s_{n_k}x)\cos(\tilde{s}_{n_k}x)\sin^2 \alpha + O\left( \frac{1}{n_k} \right), & \alpha \neq 0, \\
\frac{1}{n_k^2 \pi^2} \sin(s_{n_k}x)\sin(\tilde{s}_{n_k}x) + O\left( \frac{1}{n_k^2} \right), & \alpha = 0,
\end{cases}
\]
\[
= \begin{cases} 
\frac{1}{2} \left( 1 + \cos(2s_{n_k}x) \right) \sin^2 \alpha + O\left( \frac{1}{n_k^2} \right), & \alpha \neq 0, \\
\frac{1}{2n_k^2 \pi^2} \left( 1 - \cos(2s_{n_k}x) \right) + O\left( \frac{1}{n_k^3} \right), & \alpha = 0.
\end{cases}
\]

So, letting \( k \to \infty \) in (2.8), we have
\[
\int_0^x (q - \overline{q} - C) = 0.
\]

Therefore \( q = \overline{q} + C \) a.e. on \( (0, b) \).

Finally, when \( k \) is large, consider the Sturm–Liouville problems on the subinterval between the twin nodal point \( (x^{(n_k)}_{j_k}, x^{(n_k)}_{j_k+1}) \)
\[-y'' + qy = \lambda y \quad \text{and} \quad -y'' + (\overline{q} + C)y = (\lambda + C)y\]
such that
\[
y(x^{(n_k)}_{j_k}) = y(x^{(n_k)}_{j_k+1}) = 0.
\]
By the uniqueness of the first eigenvalue of this problem, $\lambda_{n_k} = \overline{\lambda}_{n_k} + C$.

**Remark.** The latter part of the above proof is due to Yang. It would be interesting to know if there is an alternative method other than the Lagrange identity.

Define $BA$ to be $T$-dense on $(0,b)$ if $B(A)$ is twin and for all $x \in (0,b)$, there is a subsequence of twin nodal points $x_{i_k}^{(n_k)}$ and $x_{i_k+1}^{(n_k)}$ convergent to $x$. In this case, $\overline{i}_{i_k}^{(n_k)}$ is well defined; we can use the reconstruction formula (2.4) or (2.6) to show directly the uniqueness of the potential function on $(0,b)$.

The proof of the theorem below follows the same line as the above proof, and will be omitted.

**Corollary 2.5.** Take $0 < b \leq 1$. Suppose $B(A)$ is twin and dense on $(0,b)$. If for all $k \geq 1$,

$$x_{i_k+j}^{(n_k)} = \overline{x}_{i_k+j}^{(n_k)},$$

for all $j \in \mathbb{Z}$ such that $x_{i_k+j}^{(n_k)} \in B(A)$, then

$$\alpha = \overline{\alpha}, q = \overline{q} + C \quad \text{and} \quad \lambda_n = \overline{\lambda}_n + C,$$

for all $n \geq 1$. Here $C = (\text{scot } \beta - \text{scot } \overline{\beta}) + j_1^2 (q - \overline{q})$.

### 3. Applications

In this section, we investigate several applications of Theorem 1.1. Hereafter in this paper, we consider two Sturm–Liouville problems defined by $(q, \alpha, \beta)$ and $(\overline{q}, \overline{\alpha}, \overline{\beta})$ in $L^1(0,1) \times [0, \pi)^2$ such that $\int_0^1 q = \int_0^1 \overline{q} = 0$. We shall explore sufficient conditions to obtain $q = \overline{q}$ a.e. on $(0,1)$.

The first case is an analogue of a theorem of Hochstadt and Lieberman which states that if $\lambda_n = \overline{\lambda}_n$ for all $n \in \mathbb{N}$ and $q = \overline{q}$ a.e. on $(1/2,1)$, then $q = \overline{q}$ a.e. on $(0,1)$.

**Theorem 3.1.** Take $0 < b < 1$. Assume $\beta = \overline{\beta}$, $q = \overline{q}$ a.e. on $(b,1)$. Suppose $B(A)$ (formed from any $S$) is twin and dense on $(0,b)$. If for any sufficiently large $k$, there is some $(\overline{i}_k, \overline{n}_k)$ such that

$$x_{i_k+j}^{(n_k)} = \overline{x}_{i_k+j}^{(n_k)},$$

for all $j \in \mathbb{Z}$, such that $x_{i_k+j}^{(n_k)} \in B(A)$, then

$$(q, \alpha) = (\overline{q}, \overline{\alpha}) \quad \text{in } L^1(0,1) \times [0, \pi).$$
Proof. By Theorem 1.1, \( \alpha = \overline{\alpha} \) and \( q = \overline{q} + C \), where \( C = 2(\cot \beta - \cot \overline{\beta}) + j_0^2(q - \overline{q}) \). Now \( \beta = \overline{\beta} \), and \( j_0^2q = j_0^2\overline{q} = 0 \). Therefore \( q = \overline{q} \) a.e. \( \blacksquare \)

The second application is an improvement of Yang’s theorem.

**Theorem 3.2.** Let \( 1/2 < b \leq 1 \) and \( 0 < \varepsilon < 2b - 1 \). Assume \( B(A) \) is twin and dense on \((0, b)\) and for sufficiently large \( n \),

\[
\#\{n_k : n_k \leq n\} \geq (1 - \varepsilon)n + \frac{3\varepsilon}{2}.
\]

If for any sufficiently large \( k \in \mathbb{N} \), there is \((j_k, \mu_k)\) such that

\[
x_j^{(n_k)} = \overline{x}_j^{(n_k)},
\]

for all \( j \in \mathbb{Z} \) such that \((j_k + j, n_k) \in A\), then

\[
(q, \alpha, \beta) = (q, \overline{\alpha}, \overline{\beta}) \quad \text{in } L^1(0, 1) \times [0, \pi)^2.
\]

Yang’s theorem requires \( B(A) \) to be \( S \)-dense instead of dense. Our proof is the same as that of [15, Theorem 2.1] building on Theorem 1.1 and using an inverse spectral theorem by Gesztesy and Simon [3, Theorem 1.3], which says that knowledge of \( q \) on \([0, \frac{1}{2} + \frac{\delta}{2}]\) for some \( \delta \in (0, 1), \alpha\), and a set \( S \subset \mathbb{N} \) such that

\[
\#\{n \in S : \lambda_n \leq \lambda_0\} \geq (1 - \delta)\#\{n \in \mathbb{N} : \lambda \leq \lambda_0\} + \frac{\delta}{2},
\]

for all sufficiently large \( \lambda_0 \in \mathbb{R} \), uniquely determine \( \beta \) and \( q \) on all of \((0, 1)\).

A third application makes use of Borg’s theorem, which says that if \( q \) and \( \overline{q} \) in \( L^1(0, 1) \) correspond to the same set of eigenvalues for two different boundary conditions, then \( q = \overline{q} \) a.e.

**Theorem 3.3.** Take any \( b \in (0, 1] \) and \( S = \mathbb{N} \setminus \{1\} \). Assume that the subnodal sets \( B(A_1) \) and \( B(A_2) \) thus formed (corresponding to \((q, \alpha, \beta)\) and \((q, \alpha_2, \beta_2)\), respectively) are both twin and dense on \((0, b)\). If for any index \((j, n)\), \( x_j^{(n)} \in B(A_1) \) implies \( x_j^{(n)} = \overline{x}_j^{(n)} \) and \( y_j^{(n)} \in B(A_2) \) implies \( y_j^{(n)} = \overline{y}_j^{(n)} \) (when \( x_j^{(n)} \) and \( y_j^{(n)} \) are nodal points of \( q \) corresponding to \((\alpha, \beta)\) and \((\alpha_2, \beta_2)\)), then \( q = \overline{q} \) a.e. in \((0, 1)\).

Proof. Let \( \lambda_n (\overline{\lambda}_n) \) and \( \mu_n (\overline{\mu}_n) \) be the eigenvalues for \( q (\overline{q}) \) corresponding to the boundary conditions \((\alpha, \beta)\) and \((\alpha_2, \beta_2)\), respectively. Since \( j_0^2q = j_0^2\overline{q} = 0 \), and the boundary conditions are the same, we have, from Corollary 2.5, \( \lambda_n = \overline{\lambda}_n \) and \( \mu_n = \overline{\mu}_n \), for all \( n \in \mathbb{N} \). Thus by Borg’s theorem, \( q = \overline{q} \) a.e. on \((0, 1)\). \( \blacksquare \)
We note that in above theorem, the interval \((0, b)\) can be arbitrarily small, unlike Theorem 3.2. Similarly, we also have an inverse nodal theorem applying another inverse spectral theorem of Gesztesy and Simon [12], which says that knowledge of \(q\) on \((3/4, 1)\) plus one full set of eigenvalues for some boundary conditions and half the set of eigenvalues for some other boundary conditions determine \(q\) uniquely.

It would be interesting to obtain reconstruction formulas for \((q, \alpha, \beta)\) for the various inverse nodal problems discussed above.

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