Inverse nodal problem and Ambarzumyan problem for the $p$-Laplacian

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Abstract

We study the issues of reconstruction and stability issues of the inverse nodal problem for the one-dimensional $p$-Laplacian eigenvalue problem. A key step is the application of a modified Prüfer substitution to derive a detailed asymptotic expansion for the eigenvalues and nodal lengths. Two associated Ambarzumyan problems are also solved.

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1 Introduction

Recently there is a lot of interest in the study of the $p$-Laplacian eigenvalue problem

$$-\Delta_p y + q|y|^{p-2}y = \lambda|y|^{p-2}y,$$

$$y|_{\partial\Omega} = 0,$$

where $p > 1$, and $q \in L^2(\Omega)$. This is a quasilinear partial differential equation but many of its properties are analogous to the linear case, when $p = 2$. For example, the variational eigenvalues are associated with the $p$-energy functional $\int_{\Omega} |\nabla u|^p ds / \int_{\Omega} |u|^p dx$, and a version of Courant nodal domain theorem also holds [11]. However, not all properties are analogues of the 2-Laplacian case. Say, for periodic boundary value problem, there exist some nonvariational eigenvalues [3]. Also Fredholm alternatives may not hold, even for the one-dimensional case [10].

For the one-dimensional case, the problem becomes after scaling

$$-\left((u^{(p-1)})^\prime\right)^\prime = (p - 1)(\lambda - q(x))u^{(p-1)},$$

$$u(0) = u(1) = 0,$$

(1.1)

(1.2)

where $p > 1$, $f^{(p-1)} := |f|^{p-1}\text{sgn}f$ and $q \in L^2(0,1)$. A generalized Prüfer substitution helps to establish [2, 19] the classical Sturm-Liouville properties: the existence of countably infinite real and simple eigenvalues whose associated eigenfunctions $u_n$ has exactly $n - 1$ zeros in $(0,1)$. On the interval $[0, \infty)$, the limit point theory seems to be valid. In particular, when $q(x) \to \infty$ as $x \to \infty$, there is a sequence $\{\lambda_n\}$ tending to infinity such that the associated solution $u_n$ lies in $L^p(0,\infty)$ and $u_n$ has exactly $n - 1$ zeros [4, 5].

In this paper, we plan to investigate the solvability of some inverse problems in the one-dimensional case: inverse nodal problem and Ambarzumyan problem.

The inverse nodal problem for the classical Sturm-Liouville operator is now quite well
understood. Using the nodal set as data, some issues of uniqueness, reconstruction and stability of any potential function in $L^1(0,1)$ have been solved [18, 13, 20, 14, 16]. Let $\{x_i^{(n)}\}_{i=1}^{n-1}$ be the zeros of $u_n(x)$, and denote the nodal set $X_n = \{x_i^{(n)}\}_{i=1}^{n-1}$. Define the nodal length

$$\ell_i^{(n)} = x_{i+1}^{(n)} - x_i^{(n)}$$

for $i = 1, \ldots, n - 1$. The following results were proved in [14] and [1]. (See also [7, 8].)

**Theorem 1.1** When $p = 2$.

(a) Given the nodal set $X_n$, the potential function $q$ in $L^1$ for the Dirichlet problem (1.1) and (1.2) can be reconstructed by the following formula

$$q(x) = \lim_{n \to \infty} 2n^2 \pi^2 (n\ell_j^{(n)} - 1) + n\ell_j^{(n)} \int_0^1 q(x) \, dx,$$

where $j = j_n(x) := \max\{k : x_k^{(n)} \leq x\}$.

(b) If the Neumann eigenvalues of equation (1.1) are given by $\mu_n = (n-1)^2 \pi^2$, $n \in \mathbb{N}$, then $q = 0$ in $L^1$. If the Dirichlet eigenvalues are $\lambda_n = (n\pi)^2$ and $\int_0^1 q(x) \cos(2x) \, dx = 0$, then $q = 0$.

Part (b) above is called Ambarzumyan Theorem for the classical Sturm-Liouville operator. Recently Chakravarty and Acharyya [6] generalized it to a $2 \times 2$ vectorial Sturm-Liouville system. In 1997, Chern and Shen [9] extended it to any $n$-dimensional vectorial Sturm-Liouville system. Chern et al [8] managed to solve the Ambarzumyan problem for Sturm-Liouville operator (scalar and vectorial) for more general separated boundary conditions, in particular Dirichlet boundary conditions, with one additional assumption on $q$.

We shall look for a $p$-analogue of the above theorem. For this, we need to introduce a generalized sine function $S_p$, which is the solution of the initial value problem

$$-(S_p^{(p-1)})' = (p-1)S_p^{(p-1)}, \quad (1.3)$$

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The functions $S_p$ and $S'_p$ are in fact periodic functions ([12], see also [17]) satisfying the identity:

$$|S_p(x)|^p + |S'_p(x)|^p = 1,$$

for any $x \in \mathbb{R}$. The functions $S_p$ and $S'_p$ are in fact $p$-analogues of sine and cosine functions in the classical case. It is well known that $\hat{\pi} = 2\frac{\pi/p}{\sin(\pi/p)}$ is the first zero of $S_p$. We shall develop some further properties of $S_p$ in order to derive a more detailed eigenvalue asymptotics. This asymptotics is crucial in the solution of our problems. Our main theorems are as follows.

**Theorem 1.2** For the Dirichlet problem (1.1) and (1.2) with $q \in C^1([0,1])$, $F_n$ converges to $q$ pointwisely and in $L^1(0,1)$, where

$$F_n(x) := p(n\hat{\pi})^p(n\ell_j^{(n)} - 1) + n\ell_j^{(n)} \int_0^1 q(t)dt.$$

Define the space $\Omega$ and the space of all nodal sequences $\Sigma$ by

$$\Omega = \{ q \in C^1([0,1]) : \int_0^1 q(t)dt = 0 \},$$

$$\Sigma = \{ X = \{ x_k^{(n)} \} : X \text{ is the nodal set associated with some } q \in \Omega \}.$$

We shall see that when equipped with some suitable metrics, $\Omega$ and $\Sigma$ are homeomorphic to each other (cf. Theorem 4.3). Hence when $\overline{X}$ is the nodal set associated with $\overline{q}$, and $\overline{X}$ is close to $X$ in $\Sigma$, then $\overline{q}$ is close to $q$ in $\Omega$. That is, the inverse nodal problem is stable.

Finally we study the Ambarzumyan problem.

**Theorem 1.3** Let $q \in C^1([0,1])$. If the Neumann eigenvalues for the equation (1.1) are $\mu_n = (n-1)^p\hat{\pi}^p$ ($n \in \mathbb{N}$), then $q = 0$ on $(0,1)$.

In section 2, we shall derive the eigenvalue asymptotics with the help of a modified Prüfer substitution. In section 3, we shall prove Theorem 1.2 and solve the inverse nodal problem.
In section 3, we shall define the metrics and prove Theorem 4.3. Finally in section 4, we shall prove Theorem 1.3. We shall solve the problem for Dirichlet problem also.

Our results only work for $C^1$ potentials. It would be desirable to extend them to work for more general potentials.

2 Eigenvalue asymptotics

To start with, we study the properties of $S_p$.

Lemma 2.1  (a) Whenever $S'_p \neq 0$, $(S'_p)' = -\frac{S_p}{S'_p}p - 2S_p$;

(b) $(S_pS_p'^{(p-1)})' = |S_p'|^p - (p - 1)|S_p|^p = 1 - p|S_p|^p = (1 - p) + p|S'_p|^p$.

Proof. Part (a) follows easily from (1.3). For (b), by (1.3),

$$(S_pS_p'^{(p-1)})' = S_p'S_p'^{(p-1)} + S_p(S_p'^{(p-1)})' = |S_p'|^p - (p - 1)|S_p|^p.$$ 

The last two equalities in (b) follows from (1.4). The proof is complete.

Note that when $p = 2$, part (b) becomes

$$\cos^2 x - \sin^2 x = \frac{1}{2}(\sin 2x)' = \cos 2x,$$

which is a familiar double angle formula.

Next, we define a modified Prüfer substitution.

$$u(x) = r(x)S_p(\lambda^{1/p}\theta(x)), \quad u'(x) = \lambda^{1/p}r(x)S'_p(\lambda^{1/p}\theta(x)).$$  \hspace{1cm} (2.1)

i.e.,

$$\frac{u'(x)}{u(x)} = \frac{\lambda^{1/p}S'_p(\lambda^{1/p}\theta(x))}{S_p(\lambda^{1/p}\theta(x))}.$$
Differentiating the above equation with respect to $x$ and applying Lemma 2.1, one obtains

$$\theta' = 1 - \frac{q}{\lambda}|S_p(\lambda^{1/p}\theta(x))|^p.$$  \hspace{1cm} (2.2)

That is, for sufficiently large $\lambda$,

$$\theta' = 1 - O\left(\frac{1}{\lambda}\right), \quad \theta'' = O\left(\frac{1}{\lambda^{1-1/p}}\right).\hspace{1cm} (2.3)$$

Now we are ready to establish the basic asymptotics for eigenvalues.

**Theorem 2.2** The eigenvalues $\lambda_n$ of the Dirichlet problem (1.1), (1.2) satisfy

$$\lambda_n^{1/p} = n\hat{\pi} + \frac{1}{p\lambda_n}\int_0^1 q(t)dt + O\left(\frac{1}{\lambda_n}\right)$$

$$= n\hat{\pi} + \frac{1}{p(n\hat{\pi})^{p-1}}\int_0^1 q(t)dt + O\left(\frac{1}{n^p}\right)\hspace{1cm} (2.4)$$

as $n \to \infty$.

**Remark.** In [2], various estimates were given for more general problems, but they do not include (2.4).

**Proof.** For this problem, let $\lambda = \lambda_n$, $\theta(0) = 0$. Then $\theta(1) = \frac{n\hat{\pi}}{\lambda_n}$. Integrating both sides of (2.2) over $[0, 1]$ and applying the identity

$$\frac{d}{dt}[S_p(\lambda_n^{1/p}\theta(t))S'_p(\lambda_n^{1/p}\theta(t))^{(p-1)}] = (1 - p|S_p(\lambda_n^{1/p}\theta(t))|^p)\lambda_n^{1/p}\theta'(t),$$

from Lemma 2.1(b), we obtain

$$\frac{n\hat{\pi}}{\lambda_n^{1/p}} = 1 - \frac{1}{p\lambda_n}\int_0^1 q(t)dt + \frac{1}{p\lambda_n} \int_0^1 \frac{q(t)}{\lambda_n^{1/p}\theta'} dt \left(S_p(\lambda_n^{1/p}\theta(t))S'_p(\lambda_n^{1/p}\theta(t))^{(p-1)} \right) dt. \hspace{1cm} (2.5)$$

Write $F(\lambda_n^{1/p}\theta(x)) = S_p(\lambda_n^{1/p}\theta(x))S'_p(\lambda_n^{1/p}\theta(x))^{(p-1)}$. Note that $F(\lambda_n^{1/p}\theta(x)) = 0$ when $x = 0, 1$. Hence from (2.5), we have by (2.3),

$$\int_0^1 \frac{q(t)}{\lambda_n^{1/p}\theta'} \frac{d}{dt}(F(\lambda_n^{1/p}\theta(t))) dt = \left[\frac{q(x)}{\lambda_n^{1/p}\theta'} F(\lambda_n^{1/p}\theta(x))\right]_0^1 - \int_0^1 \frac{d}{dt}\left(\frac{q(t)}{\lambda_n^{1/p}\theta'}\right) F(\lambda_n^{1/p}\theta(t)) dt$$

$$\quad = -\lambda_n^{-1/p} \int_0^1 \frac{(q(t)}{\theta'} \frac{d}{dt} F(\lambda_n^{1/p}\theta(t)) dt$$

$$\quad = O(\lambda_n^{-1/p}).$$  \hspace{1cm} (2.6)
Thus by (2.5), as $n \to \infty$,

$$
\lambda_n^{1/p} = n\hat{\pi}(1 - \frac{1}{p}\lambda_n \int_0^1 q(t) dt + O(\lambda_n^{-1+1/p}))^{-1}.
$$

Thus (2.4) follows.

\[ \square \]

**Remark.** Hence the asymptotics for $\lambda_n$ is

$$
\lambda_n = (n\hat{\pi})^p + \int_0^1 q(t) dt + O\left(\frac{1}{n}\right).
$$

## 3 The inverse nodal problem

In this section, we derive the asymptotics for nodal lengths and deduce a reconstruction formula for the function $q$. Denote that $\ell_j^{(n)} = x_{j+1}^{(n)} - x_j^{(n)}$, $1 \leq j \leq n - 1$, to be the nodal lengths of $u(x; \lambda_n)$ where $x_n^{(n)} = 1$.

**Theorem 3.1**

$$
\ell_j^{(n)} = \frac{\hat{\pi}}{\lambda_n^{1/p}} + \frac{1}{p\lambda_n} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) dt + O\left(\frac{1}{\lambda_n^{1+2/p}}\right)
$$

$$
= \frac{1}{n} + \frac{1}{p(n\hat{\pi})^p} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) dt - \frac{1}{p n (n\hat{\pi})^p} \int_0^1 q + O\left(\frac{1}{n^{p+1}}\right) \quad (3.1)
$$

as $n \to \infty$.

**Proof.** For sufficiently large $n \in \mathbb{N}$, we integrate (2.2) over $[x_j^{(n)}, x_{j+1}^{(n)}]$ and then

$$
\frac{\hat{\pi}}{\lambda_n^{1/p}} = \ell_j^{(n)} - \frac{1}{\lambda_n} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) |S_p(\lambda_n^{1/p} \theta(t))|^p dt
$$

$$
= \ell_j^{(n)} - \frac{1}{p\lambda_n} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) dt + \frac{1}{p\lambda_n} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \frac{q(t)}{\lambda_n^{1/p} \theta'} (S_p S_p^{(p-1)})' dt,
$$
by Lemma 2.1. Let $F(\tau) = S_p(\tau)S'_p(\tau)^{(p-1)}$ where $\tau = \lambda_1^{1/p} \theta(x)$. Similar to (2.6), we have

\[
\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \frac{q(t)}{\lambda_n^{1/p} \theta'} (F(\lambda_n^{1/p} \theta(t)))' \, dt = - \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \frac{q(t)}{\lambda_n^{1/p} \theta'} F(\lambda_n^{1/p} \theta(t)) \, dt \\
= - \int_{j^{(n)}}^{(j+1)^{(n)}} \left( \frac{q}{\lambda_n^{1/p} \theta'} \right)' F(\tau) \frac{d\tau}{\lambda_n^{1/p} \theta'} \\
= O\left( \frac{1}{\lambda_2^{2/p}} \right).
\]

Now we can go back to (3.2), and get the desired result easily.

\[\square\]

The above theorem shows that the nodal set $X$ is dense in $(0, 1)$. Furthermore,

\[
\frac{\lambda_n^{1/p} \ell_j^{(n)}}{\hat{\pi}} = 1 + O\left( \frac{1}{n^p} \right),
\]

uniformly for $j = 1, \ldots, n - 1$.

**Theorem 3.2** The function $q$ is given by

\[
q(x) = \lim_{n \to \infty} p\lambda_n \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\hat{\pi}} - 1 \right),
\]

for a.e. $x \in (0, 1)$, where $j = j_n(x) := \max\{k : x_k^{(n)} \leq x\}$.

**Proof.** By mean value theorem for integrals in (3.1), with fixed $n$, for each $j$, there exists $\xi \in (x_j^{(n)}, x_{j+1}^{(n)})$ such that

\[
\ell_j^{(n)} = \frac{\hat{\pi}}{\lambda_n^{1/p}} + \frac{q(\xi)}{p\lambda_n} \ell_j^{(n)} + O\left( \frac{1}{\lambda_n^{1+2/p}} \right)
\]

as $n \to \infty$. Hence

\[
q(\xi) = p\lambda_n \left( \frac{\hat{\pi}}{\lambda_n^{1/p} \ell_j^{(n)}} \right) \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\hat{\pi}} - 1 \right) + O\left( \frac{1}{\lambda_n^{1/p}} \right).
\]

Let $n \to \infty$. The proof is complete.

\[\square\]
Remarks.

1. One can also apply the Sturm-Liouville comparison theorem [2] to prove this theorem. The proof is analogous to that for Sturm-Liouville operator in [15].

2. With the asymptotic expression up to the order $\lambda_n^{1/p}$, Theorem 2.2, implies that $q(x) = \lim_{n \to \infty} F_n(x)$, where $F_n$ is determined only by the nodal data and the constant $\int_0^1 q$:

$$F_n(x) := p(n\pi)^p(n\ell_j^{(n)}) + n\ell_j^{(n)} \int_0^1 q(t) dt.$$ 

So we have proved the first part of Theorem 1.2. To show that the convergence is $L^1$, we need the following lemma.

Lemma 3.3 Suppose $q \in C([0, 1])$. Then as $n \to \infty$, with $j = j_n(x)$,

$$\| \frac{\lambda_n^{1/p}}{\pi} \int_{x_{j+1}^{(n)}}^{x_j^{(n)}} q - q(x) \|_1 \to 0.$$ 

Proof. By mean value theorem, with $x_0 \in [x, y]$, there exists $\xi \in (x, y)$ such that

$$\left| \frac{1}{y - x} \int_x^y q - q(x_0) \right| = |q(\xi) - q(x_0)|.$$ 

Due to the uniform continuity of $q$, there is a $\delta > 0$ such that the above difference is small whenever $|y - x| < \delta$. Hence, we have

$$\left| \frac{\lambda_n^{1/p}}{\pi} \int_{x_{j+1}^{(n)}}^{x_j^{(n)}} q(t) dt - q(x) \right| \leq \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} \left[ \frac{1}{x_{j+1}^{(n)} - x_j^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \right] + |q(x)| \left| \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right|.$$ 

By (3.3), we have

$$\frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} = O(1) \quad \text{and} \quad \left| \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right| = O\left( \frac{1}{n^p} \right)$$

for sufficiently large $n$. Hence, given $\varepsilon > 0$, when $n$ is large enough such that $\ell_j^{(n)} < \delta$ with $j = j_n(x)$, we have

$$\left| \frac{\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) - q(x) \right| \leq 2\varepsilon.$$
Therefore, if \( q \in C([0, 1]) \), then \( \| \frac{\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q - q(x) \|_1 \) can be arbitrarily small as \( n \) goes to infinity.

\[ \square \]

**Proof of Theorem 1.2.**

It suffices to show that the convergence is \( L^1 \). Since

\[ |F_n(x) - p\lambda_n \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right) = O \left( \frac{1}{n} \right) \]

by the asymptotic estimate of \( \lambda_n^{1/p} \), it suffices to show that as \( n \to \infty \),

\[ \| p\lambda_n \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right) - q \|_1 \to 0. \]

By (3.1), we have

\[ p\lambda_n \left( \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi} - 1 \right) = \frac{\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) dt + O \left( \frac{1}{\lambda_n^{1/p}} \right), \]

and so converges to \( q \) in \( L^1(0, 1) \) by Lemma 3.3.

\[ \square \]

## 4 Stability

In this section, we shall define the metrics on \( \Omega \) and \( \Sigma \) and study the stability of the inverse nodal problem. For \( r \geq 1 \), let \( S_n^r(X, \overline{X}) = (n\pi)^r (\sum_{k=0}^{n-1} |\ell_k^{(n)} - \overline{\ell}_k^{(n)}|)^{1/r} \). Define the metric on \( \Sigma \)

\[ d_0^r(X, \overline{X}) = \lim_{n \to \infty} S_n^r(X, \overline{X}), \]

and

\[ d_\Sigma^r(X, \overline{X}) = \lim_{n \to \infty} \frac{S_n^r(X, \overline{X})}{1 + S_n^r(X, \overline{X})}. \]
Proposition 4.1 The function $d_{\Sigma}(\cdot, \cdot)$ is a pseudometric on $\Sigma$.

Remark. Note that our definition of $d_{\Sigma}$ is similar in [16]. Thus $d_{\Sigma}(X, \overline{X}) \leq d_{0}(X, \overline{X})$. If $d_{0}(X, \overline{X}) < \infty$, then

$$d_{0}(X, \overline{X}) \leq \frac{d_{\Sigma}(X, \overline{X})}{1 - d_{\Sigma}(X, \overline{X})}.$$ 

Thus $d_{0}(X, \overline{X})$ is close to 0 iff $d_{\Sigma}(X, \overline{X})$ is close to 0. In particular, $d_{0}(X, \overline{X}) = 0$ iff $d_{\Sigma}(X, \overline{X}) = 0$.

Proof. It is obvious that $d_{\Sigma}(\cdot, \cdot)$ is finite and symmetric. To prove the triangle inequality, it suffices to show that

$$\frac{S_{n}^\tau(X, \overline{X})}{1 + S_{n}^\tau(X, \overline{X})} \leq \frac{S_{n}^\tau(X, Y)}{1 + S_{n}^\tau(X, Y)} + \frac{S_{n}^\tau(Y, \overline{X})}{1 + S_{n}^\tau(Y, \overline{X})},$$

or equivalently,

$$S_{n}^\tau(X, Y)S_{n}^\tau(Y, \overline{X})S_{n}^\tau(X, \overline{X}) + 2S_{n}^\tau(X, Y)S_{n}^\tau(Y, \overline{X}) + S_{n}^\tau(X, Y) + S_{n}^\tau(Y, \overline{X}) - S_{n}^\tau(X, \overline{X}) \geq 0.$$ 

But this is valid as from triangle inequality, $S_{n}^\tau(X, Y) + S_{n}^\tau(Y, \overline{X}) - S_{n}^\tau(X, \overline{X}) \geq 0.$

\[ \square \]

Remark. In fact $d_{\Sigma}$ is a metric on $\Sigma$. For it follows from the proof of Theorem 4.3 that $d_{0}(X, \overline{X}) = 0$ iff $q = \overline{q}$. That means $X = \overline{X}$.

Lemma 4.2 Suppose $X, \overline{X} \in \Sigma$. Then, when $n$ is sufficiently large,

(a) the interval $I_{k}^{(n)}$ between $x_{k}^{(n)}$ and $\overline{x}_{k}^{(n)}$ has length $O(\frac{1}{np})$,

(b) for all $x \in (0, 1)$, $|j_{n}(x) - \overline{j}_{n}(x)| \leq 1$. 

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Proof. (a) It follows from (3.1) that assuming \( \int_0^1 q = 0 \), the asymptotic expansion of a nodal point \( x_k^{(n)} \) is given by

\[
x_k^{(n)} = \frac{k}{n} + \frac{1}{p(n^p \pi)^p} \int_0^{x_k^{(n)}} q(t) \, dt + o(\frac{1}{n^{p+1}}) + O(\frac{1}{n^{2p-1}}).
\]

(b) Fix \( x \in (0, 1) \). Let \( j = j_n(x) \), \( \bar{j} = \bar{j}_n(x) \). Since

\[
\frac{j}{n} + O(\frac{1}{n^p}) = x_j^{(n)} \leq x \leq x_{j+1}^{(n)} = \frac{j+1}{n} + O(\frac{1}{n^p})
\]

and

\[
\frac{j}{n} + O(\frac{1}{n^p}) = x_j^{(n)} \leq x \leq x_{j+1}^{(n)} = \frac{j+1}{n} + O(\frac{1}{n^p}),
\]

when \( n \) is large enough, \( \bar{j} + 1 \geq j \) and \( j + 1 \geq \bar{j} \). Hence \(-1 \leq \bar{j} - j \leq 1\).

\[\square\]

**Theorem 4.3** For any \( r \geq 1 \), \( d_r^\Sigma \) is a metric on \( \Sigma \). Furthermore the metric space \((\Sigma, d_r^\Sigma)\) is homeomorphic to the space \( \Omega \) with the metric induced by \( \| \cdot \|_r \).

**Proof.** It suffices to show that

\[
\| q - \bar{q} \|_r = p d_0^\Sigma(X, \bar{X}).
\]

For a.e. \( x \in (0, 1) \), by Theorem 3.2, we have

\[
q(x) - \bar{q}(x) = \lim_{n \to \infty} p(n^p \pi)^p n (\ell_j^{(n)}(x) - \ell_{\bar{j}}^{(n)}(x)).
\]

Hence by Fatou’s lemma,

\[
\| q - \bar{q} \|_r \leq \limsup_{n \to \infty} p(n^p \pi)^p \| \ell_j^{(n)}(x) - \ell_{\bar{j}}^{(n)}(x) \|_r
\]

\[
\leq p n^p \limsup_{n \to \infty} (n^{p+1} \| \ell_j^{(n)}(x) - \ell_{\bar{j}}^{(n)}(x) \|_r + n^{p+1} \| \ell_{\bar{j}}^{(n)}(x) - \ell_{\bar{j}}^{(n)}(x) \|_r).
\]

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Here, by Lemma 4.2, the second term becomes
\[
n^{p+1} \|\ell_{j_n(x)}^{(n)} - \ell_{\overline{j}_n(x)}^{(n)}\|_r = n^{p+1} \left( \sum_{k=0}^{n-1} |\ell_{k+1}^{(n)} - \ell_k^{(n)}| r |\ell_k^{(n)}| \right)^{1/r}
\]

because by (3.1) and continuity of \( q \),
\[
|\ell_k^{(n)} - \overline{\ell}_k^{(n)}| = \frac{1}{p(n\pi)^{p+1}} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t) dt - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t) dt + o\left(\frac{1}{n^{p+2}}\right) + O\left(\frac{1}{n^{2p}}\right)
\]

Hence,
\[
\|q - \overline{q}\|_r \leq pn^{p+1} \pi p \lim_{n \to \infty} \left( \sum_{k=0}^{n-1} |\ell_k^{(n)} - \overline{\ell}_k^{(n)}| r \right)^{1/r}
\]

Conversely, using the above derivations, for sufficiently large \( n \),
\[
\|q - \overline{q}\|_r + o(1) = pn^{p+1} \pi p \lim_{n \to \infty} \left( \sum_{k=0}^{n-1} |\ell_k^{(n)} - \overline{\ell}_k^{(n)}| r \right)^{1/r}
\]

Hence as \( n \) tends to infinity,
\[
p d_0^r(\mathbf{X}, \mathbf{X}) \leq \|q - \overline{q}\|_r.
\]

The proof is complete.
5 Ambarzumyan problems

We first consider the Ambarzumyan problem for Dirichlet boundary conditions.

**Theorem 5.1** If the eigenvalues of the problem (1.1), (1.2) are \( \lambda_n = (n\pi)^p, n \geq 1 \), and the function \( q \) satisfies

\[
\int_0^1 q(x)(S_p(\hat{\pi}x)S_p'(\hat{\pi}x)^{(p-1)})'dx = 0, \tag{5.1}
\]

then \( q(x) = 0 \) a.e. on \((0,1)\).

**Proof.** By (2.5),

\[
\int_0^1 q(x) = 0.
\]

Next we show that \( S_p(\hat{\pi}x) \) is the first eigenfunction. By the variational principle (cf.[1]),

\[
\lambda_1 = \inf_{u \neq 0} \frac{\int_0^1 |u'|^p + (p-1) \int_0^1 q|u|^p}{(p-1) \int_0^1 |u|^p},
\]

where \( u \in C^2[0,1] \), satisfying the boundary condition (4.2). Now \( S_p(\hat{\pi}x) \) satisfies (4.2), and

\[
\hat{\pi} = \lambda_1 \leq \frac{\int_0^1 |S'_p(t)|^p dt + (p-1) \int_0^1 q(t)|S_p(t)|^p dt}{(p-1) \int_0^1 |S_p(t)|^p dt}. \tag{5.2}
\]

By Lemma 2.1(b) and the equations (5.1) and (5.2), we have,

\[
\int_0^1 |S'_p(\hat{\pi}t)|^p dt = \int_0^1 \frac{p-1}{p} dt + \frac{1}{\pi p} \int_0^1 (S_p S_p^{(p-1)})' dt = \frac{p-1}{p},
\]

\[
\int_0^1 q(t)|S_p(\hat{\pi}t)|^p dt = \frac{1}{p} \int_0^1 q(t) dt - \frac{1}{\pi p} \int_0^1 q(t)(S_p S_p^{(p-1)})' dt = 0.
\]

So in (5.2), both numerator and denominator are \((p-1)/p\). Hence \( S_p(\hat{\pi}x) \) achieves the minimum value and is thus the first eigenfunction. Substituting this into (1.1), we obtain \( q \equiv 0 \) a.e. on \((0,1)\).

\( \square \)
Finally we study the Ambarzumyan problem for Neumann boundary conditions:

\[ u'(0) = u'(1) = 0. \] (5.3)

We need the following lemma on eigenvalue asymptotics.

**Lemma 5.2** The eigenvalues \( \lambda_n \) of the problems (1.1), (5.3) satisfy

\[
\lambda_n^{1/p} = (n - 1) \hat{\pi} + \frac{1}{p(n - 1)\hat{\pi}^{p-1}} \int_0^1 q(x)dx + O\left(\frac{1}{n^p}\right),
\] (5.4)

as \( n \to \infty \).

**Proof.** The proof is similar to Theorem 2.2. From the phase equation

\[
\theta' = 1 - \frac{q}{\lambda} |S_p(\lambda^{1/p} \theta(x))|^p,
\] (5.5)

we let \( \lambda = \lambda_n \) and \( \theta(0) = \hat{\pi}/2 \). Then \( \theta(1) = (n - 1/2)\hat{\pi} \). Integrate both sides of (5.5) over \([0,1] \). Thus we obtain

\[
\frac{(n - 1)\hat{\pi}}{\lambda_n^{1/p}} = 1 - \frac{1}{p\lambda_n} \int_0^1 q(t) dt + \frac{1}{p\lambda_n} \int_0^1 q(t)(S_pS_p^{(p-1)})' dt
\]

\[
= 1 - \frac{1}{p\lambda_n} \int_0^1 q(t) dt + O(\lambda_n^{-1-\frac{1}{p}}).
\]

Thus,

\[
\lambda_n^{1/p} = (n - 1) \hat{\pi} + \frac{1}{p\lambda_n^{1-1/p}} \int_0^1 q(t) dt + O\left(\frac{1}{\lambda_n}\right)
\]

as \( n \to \infty \) and the lemma holds.

\[ \square \]

The proof of Theorem 1.3 is now ready as an analogue of Theorem 5.1. Yet we give an alternative proof using Yurko’s argument[21].

**Proof of Theorem 1.3.**
By Lemma 5.2, \( \int_0^1 q(x)dx = 0 \). Let \( u_1(x) \) be an eigenfunction associated with \( \lambda_1 = 0 \). According to Binding & Drábek[2], the eigenfunction \( u_1(x) \) has no zeros in the interval \([0, \pi]\). Taking into account the relation

\[
\left( \frac{u_1'}{u_1} \right)^{(p-1)} = \frac{(u_1^{(p-1)})'}{u_1^{(p-1)}} - (p-1)\frac{|u_1'|^p}{u_1},
\]

we get, (c.f. [21]),

\[
0 = \int_0^1 q(x)dx = \frac{1}{p-1} \int_0^1 \frac{(u_1^{(p-1)})'}{u_1^{(p-1)}}dx
= \int_0^1 \left| \frac{u_1'}{u_1} \right|^pdx.
\]

Thus \( u_1'(x) = 0 \), and \( u_1 \) is constant. So \( q(x) = 0 \).

\[\square\]

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