An initial value approach to rotationally symmetric harmonic maps

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Abstract

We study the effect of the varying \( y'(0) \) on the existence and asymptotic behavior of solutions for the initial value problem

\[
\begin{cases}
   y''(r) + (n - 1) \frac{f'(r)g'(y(r))}{f(r)} - (n - 1) \frac{g'(y(r))g(y(r))}{f(r)^2} = 0, \\
y(0) = 0,
\end{cases}
\]

where \( f \) and \( g \) are some prescribed functions. Global solutions of this ODE on \([0, \infty)\) represent rotationally symmetric harmonic maps, with possibly infinite energies, between certain class of Riemannian manifolds. By studying this ODE, we show among other things that (i) all rotationally symmetric harmonic maps from \( \mathbb{R}^n \) to the hyperbolic space \( \mathbb{H}^n \) blow up in a finite interval; (ii) all such harmonic maps from \( \mathbb{H}^n \) to \( \mathbb{R}^n \) are bounded; and (iii) a trichotomy phenomenon occurs for such harmonic maps from \( \mathbb{H}^n \) into itself, viz., they blow up in a finite interval, are the identity map, or are bounded according as the initial value \( y'(0) < 1, = 1, \) or \( > 1 \). Finally when \( n = 2 \), the above equation can be solved exactly by quadrature method. Our results supplement those of Ratto and Rigoli (J. Differential Equations 101 (1993) 15–27) and Tachikawa (Tokyo J. Math. 11 (1988) 311–316).

\( \text{Keywords: Harmonic map; Rotational symmetry; Model space; Hyperbolic space} \)

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1. Introduction

We study the existence and nonexistence of rotationally symmetric harmonic maps between certain kinds of noncompact Riemannian manifolds called “model spaces.” This involves the analysis of a nonlinear ordinary differential equation which comes from the original elliptic system for the harmonic map, when symmetries are imposed on the manifolds as well as the map itself. Following [4] we denote the domain manifold by $M^n(f)$ and the target manifold by $N^n(g)$ defined in the following way:

$$M^n(f) = \left( S^{n-1} \times [0, +\infty), f^2(\theta) d\theta^2 + dr^2, \right)$$

$$N^n(g) = \left( S^{n-1} \times [0, +\infty), g^2(\theta) d\theta^2 + dr^2, \right)$$

where $(S^{n-1}, d\theta^2)$ is the $(n-1)$-dimensional Euclidean sphere and the functions $f, g$ satisfy the conditions

\begin{align*}
  f, g &\in C^2[0, \infty), \quad f(0) = g(0) = 0, \quad f'(0) = g'(0) = 1. \quad (1.1)
\end{align*}

$M^n(f)$ and $N^n(g)$ are called model spaces (or “models” [3]). A (rotationally) symmetric harmonic map from $M^n(f)$ to $N^n(g)$ is a critical point of the functional

$$F(F) = \int_M \left\{ f^{a\beta} \frac{\partial F^i}{\partial x^a} \frac{\partial F^j}{\partial x^\beta} g_{ij} \right\} \sqrt{f} \, dx,$$

where $f^{a\beta}$ and $g_{ij}$ denote the metric of $M^n(f)$ and $N^n(g)$, respectively, $x_\alpha, y_i (\alpha, i = 1, \ldots, n)$ are the coordinates for $M^n(f)$ and $N^n(g)$ and the map $F = F(r, \theta)$ satisfies the rotational symmetry condition, i.e.,

$$F(r, \theta) = (y(r), \theta).$$

In this situation the critical points of the symmetric functional $F$ are exactly the symmetric critical points of the general functional. This is the so-called “principle of symmetric criticality” (see, for example, [1]). Thus we may compute the first variation of $F(F)$ to obtain the Euler–Lagrange equation (as shown in [5]) of the form

\begin{align*}
  y''(r) + (n-1) \frac{g(y(r)) y'(r)}{f(r)} = 0, \\
  y(0) = 0. \quad (1.2)
\end{align*}

Recently Ratto and Rigoli [4] analyzed the above problem to obtain examples of bounded harmonic maps with infinite energies between certain model spaces with prescribed asymptotic behavior. Among other things, they simplified a proof in [5] for the nonexistence of rotationally symmetric harmonic map from $\mathbb{R}^n$ to $\mathbb{H}^n$, and proved the existence of harmonic maps from $\mathbb{H}^n$ to itself with prescribed limits at infinity by a supersolution and subsolution method.

In this paper, we shall study the differential equation (2) from the point of view of an initial value problem. Rewriting this differential equation in a form more suitable for integration from the initial value and using elementary methods, we are able to recover results in [4,6] concerning the model spaces $\mathbb{R}^n$ and $\mathbb{H}^n$. In addition to this we prove that a trichotomy phenomenon holds for rotationally symmetric harmonic maps between hyperbolic spaces, i.e., a solution $y$ to the differential equation on the hyperbolic space is either
bounded, the identity, or blows up on a finite interval, according as the initial value $y'(0)$ is smaller than, equal to, or bigger than one. This harmonic map, even if bounded, carries infinite energy.

The intuition for the above result lies in the conformal invariant case $n = 2$. One easily sees that $\alpha \cdot \text{id}$ are harmonic maps from the Poincaré disk model of $\mathbb{H}^2$ to itself and that these maps are bounded, the identity map or cannot be entire depending on whether $\alpha < 1$, $= 1$, or $> 1$. In fact, the Euler–Lagrange equation can be integrated in this case and exact solutions can be found, as will be shown later. However in general when $n > 2$, $\alpha \cdot \text{id}$ are not harmonic maps for $\alpha \neq 1$. Thus the harmonic map which correspond to each $y'(0) = \alpha$ is nontrivial.

We shall concentrate on the following four cases:

(i) $M^n(f) \cong N^n(g) \cong (\text{isometric to}) \mathbb{H}^n$, i.e., $f(r) = g(r) = \sinh r$;
(ii) $M^n(f) \cong N^n(g) \cong \mathbb{R}^n$, i.e., $f(r) = g(r) = r$;
(iii) $M^n(f) \cong \mathbb{H}^n, N^n(g) \cong \mathbb{R}^n$, i.e., $f(r) = \sinh r, g(r) = r$;
(iv) $M^n(f) \cong \mathbb{R}^n, N^n(g) \cong \mathbb{H}^n$, i.e., $f(r) = r, g(r) = \sinh r$.

Case (i) will be discussed in Section 3. For case (ii), it is obvious that solutions have to be of the form $y = cr$ ($c > 0$). For case (iii), Ratto and Rigoli [4] proved that there exists some $L > 0$ such that for any $a \in [0, L]$, there is a solution to problem (1.2) satisfying $\lim_{r \to \infty} y(r) = a$. Here we show that all solutions to problem (1.2) can be extended to a bounded harmonic map on $(0, \infty)$. Our proof is simpler than that of [4, Theorem 3.3], and although our assumption is slightly different from that of [4], it holds for all model spaces mentioned in [4, Applications 3.14]. Case (iv) was first considered in [6], where it was shown that there is no global solution on $(0, \infty)$. We shall give a simple alternative proof of the fact that any solution has to blow up on a finite subinterval of $(0, \infty)$. Cases (iii) and (iv) will be discussed in Section 4. In Section 5 we shall integrate the equation directly and derive the solution when the dimension $n = 2$. In Section 2, we shall prove the existence and uniqueness of local solutions for any given $y'(0)$. Furthermore we show that any solution of problem (1.2) in $C^2(0, \epsilon] \cap C[0, \epsilon]$ has the limit $y'(0)$. Thus all the later integration from the initial point are justified.

The interested reader may consult [4] for a comparison with other geometric results and [7] for results on more general model spaces. Our initial value approach can be generalized to consider $p$-harmonic maps [2], where global existence and uniqueness results are even scarce.

Let us first derive the two differential identities from Eq. (1.2). Multiplying $f(r)^{n-1}$ to Eq. (1.2) gives

$$(f^{n-1}y')' = (n - 1)f^{n-3}g(y)g'(y).$$

On the other hand, we also have

$$f(fy')' = (n - 1)g(y)g'(y) - (n - 2)ff'y',$$

which implies

$$\{(fy')^2\}' = (n - 1)\{g(y)^2\}' - (n - 2)ff'y'^2.$$

(1.4)
2. Local existence

**Theorem 2.1.** Suppose condition (1.1) is satisfied. Then for any \( \alpha > 0 \), there is a unique solution \( y \in C^2[0, \epsilon] \) for some \( \epsilon > 0 \) of problem (1.2) such that \( y'(0) = \alpha \) and \( y > 0 \) on \( (0, \epsilon] \).

**Proof.** Eq. (1.2) is equivalent to

\[
y'' + \frac{n-1}{r} y' - \frac{n-1}{r^2} y = (n-1) \left( \frac{1}{r} - \frac{f'}{f} \right) y' + (n-1) \left( \frac{g(y)g'(y)}{f^2} - \frac{y}{r^2} \right).
\]

(2.1)

Let \( z = y/r \). Then the above equation becomes

\[
r z'' + (n+1) z' = (n-1) \left( \frac{1}{r} - \frac{f'}{f} \right)(rz'+z) + (n-1) \left( \frac{g(rz)g'(rz)}{f(r)^2} - \frac{z}{r} \right).
\]

(2.2)

For Eq. (2.2), the homogeneous part has linearly independent solutions 1 and \( r^{-n} \), with Wronskian \(-n r^{-(n+1)}\). Letting \( v(r) = \sqrt{r} z'(r) \), the second-order equation (2.2) with \( \lim_{r \to 0} z(r) = \alpha \) is equivalent to the following system of integral equations:

\[
z(r) = T_1(z, v) = \alpha + \frac{n-1}{n} \int_0^r \left( 1 - \frac{s^n}{r^n} \right) \Phi(s, v(s), z(s)) \, ds,
\]

\[
v(r) = T_2(z, v) = \int_0^r \frac{s^n}{r^{n+1/2}} \Phi(s, v(s), z(s)) \, ds,
\]

(2.3)

where

\[
\Phi(s, v, z) = \left( 1 - \frac{f'(s)}{f(s)} \right)(\sqrt{s} v + z) + \left( \frac{g(sz)g'(sz)}{f(s)^2} - \frac{z}{s} \right).
\]

Since \( \Phi(s, v, z) \) has partial derivatives in \( v \) and \( z \), we only need to make sure that all of \( \Phi, \partial \Phi/\partial v, \) and \( \partial \Phi/\partial z \) have limits at \( s = 0 \). Assume that when \( r \) is close to 0,

\[
f(r) = r + f_1 r^2 + o(r^2), \quad g(r) = r + g_1 r^2 + o(r^2)
\]

for some \( f_1, g_1 \in \mathbb{R}^n \). Then as \( s \) tends to 0,

\[
1 - \frac{f'(s)}{f(s)} = -f_1 + O(s),
\]

(2.4)

\[
A \triangleq \frac{g(sz)g'(sz)}{f(s)^2} - \frac{z}{s} = 3g_1 z^2 - 2f_1 z + o(1),
\]

(2.5)

and

\[
\frac{\partial A}{\partial z} = 6 g_1 z - 2 f_1 + o(1).
\]
Hence \( \Phi, \vartheta \Phi / \vartheta z, \) and \( \vartheta \Phi / \vartheta v \) have limits at \((0, \alpha, 0)\). So, for \( \epsilon > 0 \) sufficiently small, \( T = (T_1, T_2) \) is a contraction map on the complete metric space

\[
S = \left\{ (z, v) : z, v \in C[0, \epsilon] \text{ with } \sup_{(0, \epsilon)} |z(r) - \alpha| \leq \frac{\alpha}{2} \text{ and } \sup_{(0, \epsilon)} |v(r)| \leq 1 \right\}.
\]

Therefore by contraction mapping theorem, system (2.3) has a unique solution which is also equal to \( z \) and is equal to \( \gamma \).

As a result, \( S_i \). On the other hand, from (2.1),

\[
\left| \frac{v(r)}{\sqrt{r}} - \frac{z}{r} \right| \leq \frac{\epsilon}{\sqrt{r}}.
\]

First from (1.3), we know \( f' = 0 \), \( g \) is bounded and \( r/2 < f(r) < 3r/2 \). So by (1.3),

\[
(f^{n-1}y)' \leq M_1 r^{n-3}
\]

for some positive constant \( M_1 \). After an improper integration on \((0, r)\),

\[
y'(r) \leq \frac{M_2}{r}
\]

for some constants \( M_2 \). Thus \( ry'(r) \) is bounded on \((0, \epsilon)\). Now by Eq. (2.3),

\[
\lim_{r \downarrow 0} y'(r) = \lim_{r \downarrow 0} \left( \frac{v(r)}{\sqrt{r}} \right) = \frac{v(0)}{\sqrt{0}} = 0.
\]

Thus \( y''(0) \) exists. Therefore \( y \) is \( C^2 \) at \( r = 0 \).

Proposition 2.2. Suppose condition (1.1) is satisfied and \( f, g \in C^2[0, \epsilon] \) with \( f''(0) = g''(0) = 0 \). If \( y \in C^2[0, \epsilon] \cap C[0, \epsilon] \) is a solution to problem (1.2), then \( \lim_{r \downarrow 0} y'(r) \) exists.

Proof. First from (1.3), we know \( f^{n-1}y' \) is monotonic. Hence \( \lim_{r \downarrow 0} f(r)^{n-1}y'(r) \) exists and is equal to \( y \). Since \( y(0) = 0 \), \( y \) has to vanish. Now on \([0, \epsilon]\), \( g(y)g'(y) \) is bounded and \( r/2 < f(r) < 3r/2 \). So by (1.3),

\[
(f^{n-1}y')' \leq M_1 r^{n-3}
\]

for some positive constant \( M_1 \). After an improper integration on \((0, r)\),

\[
y'(r) \leq \frac{M_2}{r}
\]

for some constants \( M_2 \). Thus \( ry'(r) \) is bounded on \((0, \epsilon)\). Now by Eq. (2.3),

\[
\Phi_0 \equiv \lim_{s \to 0} \Phi(s, v(s), z(s)) = -f_1 \alpha + 3g_1 \alpha^2 - 2f_1 \alpha.
\]
\[ z(\epsilon) = z(r) + \frac{n - 1}{n} \int_r^\epsilon \left( 1 - \frac{s^n}{r^n} \right) \Phi(s, v(s), z(s)) \, ds, \]
\[ v(\epsilon) = v(r) + (n - 1) \int_r^\epsilon \frac{s^n}{r^{n+1}} \Phi(s, v(s), z(s)) \, ds, \]
where, since \( f_1 = g_1 = 0 \), in (2.4) and (2.5),
\[ \Phi(s, v, z) = O \left( s y'(s) + sz(s) \right). \]
So it is bounded on \((0, \epsilon)\). Thus both integrals on the right-hand side above are Cauchy as \( r \downarrow 0 \). In fact, let
\[ \psi(r) = \int_r^\epsilon \left( 1 - \frac{s^n}{r^n} \right) \Phi(s, v(s), z(s)) \, ds. \]
When \( r_1, r_2 \) both tend to 0, \(|\psi(r_2) - \psi(r_1)|\) also tends to 0. Hence \( \psi(r) \) is Cauchy as \( r \downarrow 0 \). There \( \lim_{r \downarrow 0} \psi(r) \) exists and so does \( \lim_{r \downarrow 0} z(r) \). Similarly \( \lim_{r \downarrow 0} v(r) \) also exists.

**Remark.** We note that all the \( f \)'s and \( g \)'s as specified in cases (i)–(iv) satisfy the hypothesis of Proposition 2.2. If \( y' = O(r^{-1+\eta}) \) for some \( \eta > 0 \), then restrictions on \( f \) and \( g \) may be removed. This proposition, coupled with the preceding theorem, asserts that for any solution \( y \in C^2(0, \epsilon) \cap C[0, \epsilon] \) of problem (2), \( y'(0) \) exists and thus \( y \in C^2[0, \epsilon] \).

**Proposition 2.3** [4]. Assume that \( g' > 0 \). Then \( y' > 0 \) for any nonconstant solution of (1.2).

**Proposition 2.4.** Assume that \( g'' \geq 0 \). If \( y_1 \) and \( y_2 \) are two solutions of (1.2) and \( y_1'(0) > y_2'(0) \), then \( y_1(r) > y_2(r) \) for all \( r > 0 \).

**Proof.** Let \( w = y_1 - y_2 \). Then by Eq. (1.4),
\[ (f^{n-1}w')' = (n - 1)f^{n-3} \left( g(y_1)g'(y_1) - g(y_2)g'(y_2) \right). \]
Since \( g'' \geq 0 \), we have \( gg' \) increasing. Hence as long as \( y_1 > y_2 \), \( f^{n-1}w' \) is nondecreasing in \( r \). As \( w'(0) > 0 \), \( w' \) can never drop below zero. Therefore \( w' = y_1' - y_2' > 0 \) for all \( r \geq 0 \), which implies, after an integration, \( y_1 > y_2 \) for all \( r > 0 \). □

### 3. Harmonic maps on hyperbolic spaces

In this section we study rotationally symmetric harmonic maps between \( \mathbb{H}^n \). In this case Eq. (1.2) becomes
\[ y'' + (n - 1) \frac{\cosh r}{\sinh r} y' - (n - 1) \frac{\sinh y \cosh y}{\sinh^2 r} = 0. \]
Proposition 3.1. Suppose $M^n(f)$ and $N^n(g)$ are two model spaces with $f$ and $g$ satisfying condition (1.1). If $f \equiv g$ and satisfies $f \in C^1(0, \infty)$ and $f''(0) = 0$, then

(a) $y'(0) = 1$ implies $y(r) = r$ is the unique solution;
(b) $y'(0) < 1$ implies $y(r) < r$ for all $r > 0$;
(c) $y'(0) > 1$ implies $y(r) > r$ for all $r > 0$.

Proof. Obviously if $f \equiv g$, then $y(r) = r$ is a solution to Eq. (1.3). With such $f$, $y'(0)$ exists for any solution and thus Theorem 2.1 guarantees the uniqueness of solution.

If $y'(0) < 1$, by continuity $y' < 1$ initially, and as long as $y' < 1$,

$$(f(r)^{n-1} y')' < (n-1) f(r)^{n-2} f'(r),$$

so that $y' < 1$ after an integration. Thus we have proved (b). The proof for (c) is similar. □

Lemma 3.2. Suppose $y$ is a solution of Eq. (3.1) such that $y(0) = 0$. Then if $y'(0) < 1 - \eta$ for some $\eta \in (0, 1)$, then $y'(r) \leq 1 - \eta$ for all $r \geq 0$. On the other hand, if $y'(0) > 1 + \eta$ for some $\eta > 0$, then $y'(r) \geq 1 + \eta$ for all $r \geq 0$.

Proof. To prove the first part, it suffices to show that if $y'(r_0) = 1 - \eta$, then $y''(r_0) < 0$, for that means $y'$ can never grow beyond $1 - \eta$. Suppose $y''(r_0) \geq 0$, then from Eq. (3.1),

$$(n - 1) \frac{\cosh r_0}{\sinh r_0} y'(r_0) \leq y''(r_0) + (n - 1) \frac{\cosh r_0}{\sinh r_0} y'(r_0) = (n - 1) \frac{\sinh y(r_0) \cosh y(r_0)}{\sinh^2 r_0} \leq (n - 1) \frac{\sinh(1 - \eta) r_0 \cosh(1 - \eta) r_0}{\sinh^2 r_0}.$$

Therefore

$$y'(r_0) \leq \frac{\sinh(1 - \eta) r_0}{\sinh r_0} < 1 - \eta,$$

a contradiction. Similar argument may be used to show that if $y'(0) > 1 + \eta$ for some $\eta > 0$, then $y' \geq 1 + \eta$ on $[0, \infty)$. □

Theorem 3.3. Suppose $y$ is a solution to Eq. (3.1) such that $y(0) = 0$. If $y'(0) < 1$, then $y$ is bounded on $[0, \infty)$. If $y'(0) > 1$, then $y$ blows up at some $r > 0$.

Proof. From Lemma 3.2 and Eq. (3.1),

$$(\sinh^{-n+1} y')' \leq (n - 1) \sinh(1 - \eta) r \cosh(1 - \eta) r \sinh^{-n-3} r. \quad (3.2)$$

Now by the compound angle formula for sinh function,

$$\sinh(1 - \eta) r < \frac{\sinh r}{\cosh \eta r} < 2 e^{-\eta r} \sinh r.$$
Therefore Eq. (3.2) becomes
\[(\sinh^{n-1} r y')' < 2e^{-\eta r} (\sinh^{n-1} r)'\].

Using integration by parts, we obtain
\[
I = \int_0^r e^{-\eta r} (\sinh^{n-1} r)' dr = e^{-\eta r} \sinh^{n-1} r + \eta \int_0^r e^{-\eta r} \sinh^{n-1} r \, dr
\]
\[\leq e^{-\eta r} \sinh^{n-1} r + \frac{\eta}{n-1} I.
\]
Hence
\[
I \leq \frac{n-1}{n-1-\eta} e^{-\eta r} \sinh^{n-1} r.
\] (3.3)

Therefore for all \(r > 0\),
\[y'(r) \leq \frac{2(n-1)}{n-1-\eta} e^{-\eta r},
\]
and
\[y(r) \leq \frac{2(n-1)}{\eta(n-1-\eta)} (1 - e^{-\eta r}),
\]
which is bounded on \([0, \infty)\).

On the other hand, if \(y'(0) > 1\), then there exists \(\eta > 0\) such that \(y' \geq 1 + \eta\) and so \(y \geq (1 + \eta) r\) for any \(r \geq 0\). Since
\[
cosh(1 + \eta)r > \cosh r \cosh \eta r > \frac{1}{2} e^{\eta r} \cosh r,
\]
we obtain
\[(\sinh^{n-1} r y')' \geq \frac{1}{2} e^{\eta r} (\sinh^{n-1} r)'.\]

Therefore by a similar derivation as in (3.3),
\[y'(r) \geq \frac{n-1}{2(n-1-\eta)} e^{\eta r}
\]
and
\[y(r) \geq \frac{n-1}{2\eta(n-1-\eta)} (e^{\eta r} - 1).
\]

So \(y\) is eventually greater than \(8r\). Hence from Eq. (3.1), we deduce that
\[y'' + 2(n-1)y' \geq (n-1) \exp(3y/2)
\] (3.4)
eventually. We want to show that eventually, say, when \(r \geq r_0\),
\[y' \geq e^{y/2}.
\] (3.5)

Thus
\[
2 \left[ \exp\left( -\frac{1}{2} y(r_0) \right) - \exp\left( -\frac{1}{2} y(r) \right) \right] \geq r - r_0,
\]
which implies that \(y\) has to blow up at some finite point \(r > r_0\).
To see that Eq. (3.5) holds eventually, we observe that if $y'' \geq 0$ eventually, say, for $r \geq r_1 > 0$, then $y'$ is increasing, so that
\[ \int_{r_1}^{r} y'(s)^2 \, ds \leq (r - r_1)y'(r)^2. \]
It follows from Eq. (3.4) that
\[ y'(r)^2 + 2(n - 1) \int_{r_1}^{r} y'(s)^2 \, ds \geq \frac{2(n - 1)}{3} \exp(3y/2) + \text{constant}. \]
Hence
\[ 2n(r - r_1)y'(r)^2 \geq \frac{2(n - 1)}{3} \exp(3y/2) + \text{constant}, \]
and so eventually Eq. (3.5) holds. If, however, $y'' < 0$ in some interval eventually, then $y' > (1/2)\exp(3y/2)$ in that interval. Suppose that $y' < \exp(y/2)$ afterwards and $r_2$ is the critical point (large enough), i.e., $y'(r_2) = \exp(y(r_2)/2)$. Then viewing $y'$ as a function of $y$, we denote $p(y) = y'(r)$, so
\[ p \frac{dp}{dy} + 2(n - 1)p \geq (n - 1)\exp(3y/2), \]
so that
\[ p \frac{dp}{dy} \geq (n - 1)\exp(3y/2) - 2(n - 1)\exp(y/2) \geq e^y. \]
After an integration from $y_2$ to $y$, we obtain
\[ y^2 = p^2 \geq e^y \]
for any $y > y_2$, a contradiction. We conclude that $y' > e^{y/2}$ eventually. The proof is complete. $\square$

Next using a shooting type argument, one can prove the following existence and uniqueness theorem (cf. [4, Corollary 3.16]). Denote by $y_\alpha$ the solution of Eq. (1.3) such that $y_\alpha(0) = 0$ and $y_\alpha'(0) = \alpha$.

**Theorem 3.4.** For any $a \in (0, \infty)$, there is a unique $\alpha \in (0, 1)$ such that $y_\alpha(\infty) = a$.

**Proof.** Let
\[ S_1 = \{ \alpha \in [0, 1] \mid y_\alpha(\infty) > a \}, \quad S_2 = \{ \alpha \in [0, 1] \mid y_\alpha(\infty) < a \}. \]
Obviously $0 \in S_2 \neq \emptyset$ and $1 \in S_1 \neq \emptyset$. We want to show that both $S_1$ and $S_2$ are open so that by connectedness of $[0, 1]$, there is $\alpha_0 \in [0, 1] \setminus (S_1 \cup S_2)$, and $y_{\alpha_0}(\infty) = a$.

Take $\alpha_1 \in S_1$. Since $y_{\alpha_1}$ is strictly monotonically increasing, there is some $r_1$ such that $y_{\alpha_1}(r_1) = a$. Thus $y_{\alpha_1}(r_1 + 1) > a$. By continuity with respect to initial conditions, for any $\alpha$ close to $\alpha_1$, $y_{\alpha}(r_1 + 1) > a$ so that $y_{\alpha}(\infty) > a$. Thus $S_1$ is open.
On the other hand, take $\alpha_2 \in S_2$ and let $y(\alpha_2) = a_2 < a$. Choose some $r_2$ sufficiently large so that $y_\alpha(r_2) < a_2$ for any $\alpha$ close to $\alpha_2$, with $\alpha \in (\alpha_2, (1 + \alpha_2)/2)$. By an estimate in Theorem 3.3, when $r \geq r_2$,

$$y_\alpha'(r) \leq \frac{2(n-1)}{n-2+\alpha} \exp \left\{ -(1-\alpha)r \right\} < \frac{2(n-1)}{n-2} \exp \left\{ -\frac{1}{2} (1-\alpha_2)r \right\}. $$

Therefore

$$y_\alpha(\infty) \leq a_2 + \int_{r_0}^{\infty} y_\alpha'(r) \, dr < a_2 + \frac{4(n-1)}{(n-2)(1-\alpha_2)} \exp \left\{ -\frac{1}{2} (1-\alpha_2)r_2 \right\} < a.$$

Hence $S_2$ is also open.

Uniqueness part follows directly from Proposition 2.4. □

4. Harmonic maps from $\mathbb{H}^n$ to $\mathbb{R}^n$ and vice versa

The theorem below implies that any rotationally symmetric harmonic map from $\mathbb{H}^n$ to $\mathbb{R}^n$ can be extended to a bounded map on $[0, \infty)$.

**Theorem 4.1.** Suppose there exist $a, b, r_0 > 0$ such that

$$f(r) > br^\delta \quad (\delta \geq 1), \quad f'(r) > 0, \quad \text{and} \quad g(r) \leq ar$$

for all $r \geq r_0$. Then any solution of problem (1.2) can be extended to a bounded solution on $[0, \infty)$.

**Proof.** Suppose $y$ is a global solution of problem (1.2). Then either $y$ is always less than $r_0$, or $y(r) > r_0$ starting from some $r = r_1$. Now if the latter is true, then from Eq. (1.4),

$$\left\{ (y')^2 \right\}' \leq (n-1) \left\{ g(y)^2 \right\}'. $$

Thus for all $r > \max\{r_0, r_1\}$,

$$br^\delta y' \leq a\sqrt{n-1} y,$$

so that for some positive constant $C_1$,

$$\left[ \exp(C_1 r^{1-\delta}) y \right]' \leq 0.$$

Thus after an integration,

$$y(r) \leq C_2 \exp(C_1 r^{1-\delta}) \leq C_2.$$

It remains to show that any local solution on $[0, \epsilon]$ can be extended to $[0, \infty)$. But it is easy as on any compact interval $I$, with $m = \min_{r \in I} f(r)$, any solution of Eq. (1.3) is a supersolution of

$$y'' + my' - a(n-1)y = 0.$$

Hence by weak maximum principle, $y$ has to be bounded on $I$. □
It is easy to see that the following theorem is also valid for \( f(r) \sim r^k \) \((k \in (0, 1])\). Thus it is an alternative proof to [4, Theorems 2.9, 2.12, 2.14, 2.23].

**Theorem 4.2** (Liouville type theorem). Suppose that \( f(r) = r \) and \( gg' \) is always increasing, then any nontrivial solution to problem (1.2) on \([0, \infty)\) must be unbounded.

**Proof.** If \( y \) is a nontrivial solution on \([0, \infty)\), then \( y \) is nonzero after some point \( r = r_0 \). So by Eq. (1.4),

\[
(r^{n-1}y'(r))' = (n-1)r^{n-3}g(y(r))g'(y(r)) \geq (n-1)r^{n-3}g(y(r_0))g'(y(r_0))
\]

for all \( r \geq r_0 \). Let \( C_0 = (n-1)g(y(r_0))g'(y(r_0)) \). Hence after an integration from \( r_0 \) to \( r \),

\[
y'(r) > \frac{C_0}{(n-2)r} + \frac{C_1}{r^{n-1}}
\]

\((C_1 \in \mathbb{R}^n)\), so that

\[
y(r) > \frac{C_0}{n-2} \ln r - \ln r_0 - \frac{C_1}{n} r^{-n} - y(r_0) > \frac{C_0}{2(n-2)} \ln r - \ln r_0,
\]

when \( r \) is sufficiently large. Therefore \( y \) is unbounded on \([0, \infty)\).

When \( n = 2 \), Eq. (1.4) becomes

\[
r(y'y')' = g(y(y))g'(y(y)).
\]

By the same argument,

\[
y(r) \geq C(\ln r - \ln r_1)
\]

for some \( C > 0 \) and for all \( r > r_1 \) sufficiently large. Thus \( y \) is also unbounded. \( \square \)

We now proceed to consider the blow-up behavior for case (iv),

\[
y'' + (n-1)\frac{y'}{r} - (n-1)\frac{\sinh y \cosh y}{r^2} = 0. \tag{4.1}
\]

**Theorem 4.3** (Blow up in finite interval). Suppose \( y \) is a solution of Eq. (4.1) such that \( y(0) = 0 \). Then \( y \) has to blow up at some finite \( r > 0 \).

**Proof.** When \( n = 2 \), Eq. (4.1) can be integrated directly and our theorem validated (cf. Section 5). So we may assume \( n \geq 3 \). Now by Theorem 4.2, it is easy to show that \( y > r \) eventually. And since

\[
(r^{n-1}y'(r))' = (n-1)r^{n-3}\sinh y \cosh y \geq e^r
\]

eventually, it follows that

\[
y'(r) \geq e^{r/2}.
\]

so that eventually,

\[
y(r) \geq e^{r/2} \geq 2r^2.
\]
Therefore from Eq. (4.1) we obtain, eventually,
\[ y'' + (n - 1)y' \geq (n - 1) \exp(3y/2). \]
Thus we follow the proof for Theorem 3.3 to show that \( y' \geq e^{y/2} \) eventually and so \( y \) has to blow up for some finite \( r > 0 \). □

5. Solution by quadrature when \( n = 2 \)

When \( n = 2 \), Eq. (1.4) becomes
\[ f(fy')' = g(y)g'(y). \]
Assuming \( \lim_{r \to 0} f(r)y'(r) = 0 \), we integrate the above equation on \([0, r]\) to obtain
\[ f(r)y'(r) = g(y(r)). \]  
(5.1)
Thus each individual case can be integrated as follows:

(i) Suppose \( f(r) = g(r) = r \). Then Eq. (5.1) becomes \( ry' - y = 0 \). Thus \( y = Cr \).

(ii) Suppose \( f(r) = \sinh r \) and \( g(r) = r \). Eq. (5.1) becomes
\[ y' - \text{csch} r y = 0. \]

Since
\[ \int \text{csch} r \, dr = \ln \tanh \frac{r}{2} + C, \]
we obtain
\[ y(r) = C \tanh \frac{r}{2}. \]

(iii) If \( f(r) = r \) and \( g(r) = \sinh r \), then
\[ (\text{csch} y)y' = \frac{1}{r}. \]
Therefore
\[ y(r) = 2 \tanh^{-1}(Cr). \]

(iv) If \( f(r) = g(r) = \sinh r \), then
\[ \text{csch} y'y' = \text{csch} r. \]
Therefore
\[ y(r) = 2 \tanh^{-1} \left( C \tanh \frac{r}{2} \right). \]

Since the above solutions rely on double angle formulas for hyperbolic tangent function, the same method can also solve for symmetric maps involving ellipsoids, but not further.

Finally we note that the hyperbolic tangent function maps \( \mathbb{R} \) into \([0, 1]\), thus the harmonic maps in case (ii) are necessarily bounded. On the contrary, the maps in case (iii)
have to blow up at where \( r = 1/C \). Case (iv) is most interesting, in that each map \( y \) has to be bounded, unbounded, or blow-up according as \( C < 1, = 1, \) or \( > 1 \). In fact, in this case,

\[
y'(r) = \frac{C \sech^2(r/2)}{1 + C^2 \tanh^2(r/2)},
\]

so that \( y'(0) = C \). Thus the asymptotic behavior of a symmetric harmonic map on hyperbolic spaces depends only the initial value \( y'(0) \), as indicated before.

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References