On the quasi-nodal map for the Sturm–Liouville problem

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We show that the space of Sturm–Liouville operators characterized by $H = (q, \alpha, \beta) \in L^1(0, 1) \times [0, \pi)^2$ such that $\int_0^1 q = 0$ is homeomorphic to the partition set of the space of all admissible sequences $X = \{X^{(n)}_k\}$ which form sequences that converge to $q, \alpha,$ and $\beta$ individually. This space, $\Gamma$, of quasi-nodal sequences is a superset of, and is more natural than, the space of asymptotically nodal sequences defined in Law and Tsay (On the well-posedness of the inverse nodal problem. Inv. Probl. 17 (2001), 1493–1512). The definition of $\Gamma$ relies on the $L^1$ convergence of the reconstruction formula for $q$ by the exactly nodal sequence.

1. Introduction

Consider the Sturm–Liouville operator $H$:

$$Hy = -y'' + q(x)y,$$

with boundary conditions

$$\begin{align*}
y(0) \cos \alpha + y'(0) \sin \alpha &= 0, \\
y(1) \cos \beta + y'(1) \sin \beta &= 0.
\end{align*}$$

Here $q \in L^1(0, 1)$ and $\alpha, \beta \in [0, \pi)$. Let $\lambda$ be the $n$th eigenvalue of the operator $H$ and $0 < x_1^{(n)} < x_2^{(n)} < \cdots < x_{n-1}^{(n)} < 1$ be the $(n - 1)$ nodal points of the $n$th eigenfunction. The double sequence $\{x_k^{(n)}\}$ is called the nodal sequence associated with $H$. Additionally, let $l_k^{(n)} = x_{k+1}^{(n)} - x_k^{(n)}$ be the associated nodal length. We define the function $j_n(x)$ on $(0, 1)$ by

$$j_n(x) = \max\{k : x_k^{(n)} \leq x\}.$$

Hence, if $x$ and $n$ are fixed, then $j = j_n(x)$ implies that $x \in [x_j^{(n)}, x_{j+1}^{(n)})$. 

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We identify $H$ with $(q, \alpha, \beta)$. The inverse nodal problem on the Sturm–Liouville operator, first defined by McLaughlin [10], is the problem of finding $H = (q, \alpha, \beta)$ using only the nodal set $\{x_k^{(n)}\}$. It is now known [1,10,13] that the complete operator $H$ (denoted by $(q, \alpha, \beta)$) is uniquely determined by the nodal set up to a constant $\int_0^1 q$. This constant can be eliminated if $q$ is normalized by the condition $\int_0^1 q = 0$. The reconstruction formula of $q$ has also been found [4,9] (see also [13]). The detailed reconstruction formula depends on the boundary conditions. Namely, there are four cases: (I) $\alpha = \beta = 0$; (II) $\alpha, \beta > 0$; (III) $\alpha = 0 < \beta$; (IV) $\beta = 0 < \alpha$. For each case, we define $F_n(x)$ according to the following conditions:

(a) if $\alpha = \beta = 0$, then

$$F_n(x) = 2n^2\pi^2\{n_j^{(n)} - 1\};$$

(b) if $\alpha, \beta > 0$, then

$$F_n(x) = 2(n - 1)^2\pi^2\left\{(n - 1)j_j^{(n)} - 1 + \frac{j_j^{(n)}}{(n - 1)^2}\left(\cot \beta - \cot \alpha\right)\right\};$$

(c) if $\alpha = 0 < \beta$ or $\beta = 0 < \alpha$, then

$$F_n(x) = 2(n - \frac{1}{2})^2\pi^2\left\{(n - \frac{1}{2})j_j^{(n)} - 1 + \frac{j_j^{(n)}}{n^2}\left(\cot \beta - \cot \alpha\right)\right\}.$$ 

Here $j = j_n(x)$, and

$$\cot \gamma = \begin{cases} 0, & \text{if } \gamma = 0, \\ \cot \gamma, & \text{otherwise.} \end{cases}$$

**Theorem 1.1** (see Chen et al. [2]; Law et al. [9]). $F_n$ converges to $q$ in $L^1(0,1)$ as well as pointwisely almost everywhere (a.e.).

We remark that (b) above is different from the corresponding expression in [8], where $F_n$ is expressed in terms of $n$, but not $n - 1$. The confusion stems from a basic counting lemma which should state $|s_n - (n - 1)\pi| < \frac{1}{2}\pi$ when $n$ is large. Here $s_n = \sqrt{\lambda_n}$. A proof, following the one in [11], will appear in appendix. (We remark that the lemma below was discussed in detail by Trubowitz et al. [3,5,6,11] for $q \in L^2(0,1)$.) Hence, we have the following eigenvalue asymptotics. In particular, lemma 1.2(b) corrects the corresponding statements in [8,9,13].

**Lemma 1.2.** Suppose that $(q, \alpha, \beta) \in L^1(0,1) \times [0, \pi)^2$.

(a) If $\alpha = \beta = 0$, then

$$s_n = n\pi + \frac{1}{2n\pi} \int_0^1 (1 - \cos(2n\pi t))q(t)\,dt + o\left(\frac{1}{n^2}\right).$$

(b) If $\alpha, \beta > 0$, then

$$s_n = (n - 1)\pi + \frac{1}{(n - 1)\pi} \left[\cot \beta - \cot \alpha \right.$$

$$+ \frac{1}{2} \int_0^1 (1 + \cos(2(n - 1)\pi t))q(t)\,dt] + o\left(\frac{1}{n^2}\right).$$
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(c) If $\alpha = 0 < \beta$ or $\beta = 0 < \alpha$, then

$$s_n = (n - \frac{1}{2})\pi + \frac{1}{(n - \frac{1}{2})\pi} \left[ \text{scot} \beta - \text{scot} \alpha \right] + \frac{1}{2} \int_0^1 (1 + \alpha_0 \cos((2n - 1)\pi t))q(t) \, dt + o\left( \frac{1}{n^2} \right).$$

In [8], the detailed asymptotic expansions of the nodal points $\{x_k^{(n)}\}$ made in [9] are used to define some asymptotically equivalent nodal sequence associated with some $L^1$ function $q$. This space of asymptotically equivalent nodal sequences, after a partition, was shown to be homeomorphic to the space of Sturm–Liouville operators. In this paper we enlarge the space of asymptotically equivalent nodal sequences to the space of all double sequences such that corresponding functions $F_n$ are convergent in $L^1$. These sets are called quasi-nodal to the limit operator (see definitions in § 2).

Let $N' = N \setminus \{1\}$. Define the space $\Omega$ (of Sturm–Liouville operators) and the space of all admissible sequences $\Sigma$ as follows:

$$\Omega = \left\{ (q, \alpha, \beta) \in L^1(0, 1) \times [0, \pi)^2 : \int_0^1 q = 0 \right\}$$

and $\Sigma$ is equal to the collection of all double sequences $X = \{X_k^{(n)} : k = 1, 2, \ldots, n-1; \ n \in N'\}$ such that $0 < X_1^{(n)} < X_2^{(n)} < \cdots < X_{n-1}^{(n)} < 1$ for each $n$.

Given an admissible sequence $X = \{X_k^{(n)} : k = 1, \ldots, n-1; \ n \in N'\}$, we say that $X$ is asymptotically nodal to some $H = (q, \alpha, \beta) \in \Omega$ (and say that $X$ is an asymptotically equivalent nodal sequence) if exactly one of the following cases is valid as $n \to \infty$:

(1) for any $1 \leq k \leq n - 1$,

$$X_k^{(n)} = \frac{k}{n} + \frac{1}{2n^2\pi^2} \int_0^1 (1 - \cos(2n\pi t))q(t) \, dt + o\left( \frac{1}{n^3} \right); \quad (1.2)$$

(2) there exist $A, B \in \mathbb{R}$ such that, for any $1 \leq k \leq n - 1$,

$$X_k^{(n)} = \frac{k - \frac{1}{2}}{n - 1} + \frac{1}{2(n-1)^2\pi^2} \left( \frac{1}{2} \int_0^1 X_k^{(n)} (1 + \cos(2(n-1)\pi t))q(t) \, dt - A \right)$$

$$+ \frac{k - \frac{1}{2}}{(n-1)^3\pi^2} \left( A - B - \frac{1}{2} \int_0^1 \cos(2(n-1)\pi t)q(t) \, dt \right) + o\left( \frac{1}{n^3} \right); \quad (1.3)$$
(3) there exists $B \in \mathbb{R}$ such that, for any $1 \leq k \leq n - 1$,

$$X^{(n)}_k = \frac{k}{n - \frac{1}{2}} + \frac{1}{2(n - \frac{1}{2})^2 \pi^2} \int_0^{X^{(n)}_k} \left(1 - \cos((2n - 1)\pi t)\right) q(t) \, dt$$

$$+ \frac{k}{(n - \frac{1}{2})^2 \pi^2} \left(-B + \frac{1}{2} \int_0^1 \cos((2n - 1)\pi t) q(t) \, dt + o\left(\frac{1}{n^3}\right)\right).$$

(1.4)

(4) there exists $A \in \mathbb{R}$ such that, for any $1 \leq k \leq n - 1$,

$$X^{(n)}_k = \frac{k - \frac{1}{2}}{n - \frac{1}{2}} + \frac{1}{2(n - \frac{1}{2})^2 \pi^2} \int_0^{X^{(n)}_k} \left(1 + \cos((2n - 1)\pi t)\right) q(t) \, dt - A$$

$$+ \frac{k - \frac{1}{2}}{(n - \frac{1}{2})^2 \pi^2} \left(A - \frac{1}{2} \int_0^1 \cos((2n - 1)\pi t) q(t) \, dt + o\left(\frac{1}{n^3}\right)\right).$$

(1.5)

Here all the order estimates depend only on $\|q\|_1$ and $n$.

Let $\Sigma_1 \subset \Sigma$ be the subspace of all asymptotically equivalent nodal sequences and let $\Sigma_1^* = \Sigma_1/\sim$. With a suitable metric $d_B$, Law and Tsay [8, theorem 1.3] proved the following theorem.

**Theorem 1.3** (Law and Tsay [8]). The two metric spaces $(\Omega, d_B)$ and $(\Sigma_1^*, d)_{\Sigma}$ are homeomorphic to each other.

By theorem 1.1, if $X$ is an asymptotically equivalent nodal sequence associated with $(q, \alpha, \beta) \in L^1(0, 1) \times [0, \pi)^2$, then the corresponding function $F_n$ converges to $q$ in $L^1(0, 1)$ as well as pointwisely a.e. It would be interesting to know whether the space $\Sigma_1$ can be expanded to include all sequences that converge to $q$ in some way, which seems to be a more natural space. We find that it is possible if the convergence is $L^1$ only. Define the set $\Gamma$ to be the collection of all admissible sequences that form sequences converging to some $(q, \alpha, \beta)\in \Omega$. We now state our main theorem.

**Theorem 1.4.** The metric space $(\Omega, d_B)$ is homeomorphic to the space $(\Gamma/\sim, d_{\Gamma})$ where $d_{\Gamma}$ is a pseudometric on $\Gamma$ defined in (3.1), and $\sim$ is the equivalence relation induced by $d_{\Gamma}$.

The theorem shows that the set $\Gamma$ of admissible sequences can be readily partitioned into quotient sets by the equivalence relation induced by the pseudometric $d_{\Gamma}$, i.e. $X \sim \bar{X}$ if and only if $d_{\Gamma}(X, \bar{X}) = 0$. This collection of quotient sets $\Gamma/\sim$ is homeomorphic to the space $\Omega$. Thus, the inverse nodal problem defined on $\Gamma/\sim$ is well posed.

It is clear from our analysis that, for the direct problem, the nodal set has a jump when either one or all of the boundary conditions tend to zero. For, in lemma 3.5, we show that $d_{\Gamma}(X, \bar{X}) = 1$ when $X$ and $\bar{X}$ belong to different cases of boundary conditions. We conclude that the nodal set is not continuous in $\alpha$ or $\beta$ when either one goes to zero. So, although any eigenvalue branch is a continuous function of various parameters in the Sturm–Liouville operators (including the potential function, boundary points and the boundary conditions [7]), the spectral data as a whole (for
example, the nodal set or the spectral set) may not be continuous. We shall explore on this issue further.

In general, the exactly nodal positions for the \( n \)th and \( (n + 1) \)th eigenfunctions interlace. However, in the definition of quasi-nodal sets, we ’fix’ the nodal positions for large values of \( n \), but not for small values, because we are using the asymptotic approach. It is irrelevant, at this stage, to investigate the properties of small eigenfunctions.

In §2, we give the precise definition of quasi-nodal sequences and some preliminary results. In §3 we define \( d_{\Gamma} \) and prove the main theorem. Two examples will be given in §4 to illustrate that it is the \( L^1 \) convergence, and not the pointwise convergence, that matters in the homeomorphism.

2. The space of quasi-nodal sequences

**Lemma 2.1.** Let \( X \) be an admissible sequence in \( \Sigma \). Then at most one of the following limits exists in \( \mathbb{R} \) for all \( k = 1, 2, \ldots, n - 1 \):

(i) \( \lim_{n \to \infty} n[X^{(n)}_k - k] \),

(ii) \( \lim_{n \to \infty} (n - 1)([n - 1]X^{(n)}_k - (k - \frac{1}{2})) \),

(iii) \( \lim_{n \to \infty} (n - \frac{1}{2})([n - \frac{1}{2}]X^{(n)}_k - (k - \frac{1}{2})) \).

Similarly, at most one of the following limits exists in \( \mathbb{R} \) for all \( k = 1, 2 \ldots, n - 1 \):

(iv) \( \lim_{n \to \infty} n[X^{(n)}_{n-k} - (n - k)] \),

(v) \( \lim_{n \to \infty} (n - 1)([n - 1]X^{(n)}_{n-k} - (n - k - \frac{1}{2})) \),

(vi) \( \lim_{n \to \infty} (n - \frac{1}{2})([n - \frac{1}{2}]X^{(n)}_{n-k} - (n - k)) \).

**Proof.** Since, for all fixed \( k \),

\[
X^{(n)}_k = O\left(\frac{1}{n^2}\right)
\]

and

\[
n[X^{(n)}_k - k] - (n - 1)([n - 1]X^{(n)}_k - (k - \frac{1}{2})) = (2n - 1)X^{(n)}_k - \frac{1}{2}n - (k - \frac{1}{2})
\]

does not exist as \( n \to \infty \), (i) and (ii) cannot exist at the same time.

If (i) or (ii) exists (say, the limit is \( \gamma \)), then, for all \( k = 1, 2, \ldots, n - 1 \) and \( n \) sufficiently large, we have

\[
X^{(n)}_k = \frac{k - \frac{1}{2}\delta}{n - \delta} + \frac{\gamma}{(n - \delta)^2} + o\left(\frac{1}{n^2}\right),
\]

where \( \delta = 0 \) if (i) exists and \( \delta = 1 \) if (ii) exists. Then, for all \( k = 1, 2, \ldots, n - 1 \),

\[
(n - \frac{1}{2})([n - \frac{1}{2}]X^{(n)}_k - (k - \frac{1}{2}))
\]

\[
= (n - \frac{1}{2})\left[\frac{(n - \frac{1}{2})(k - \frac{1}{2}\delta)}{n - \delta} + \frac{(n - \frac{1}{2})\gamma}{(n - \delta)^2} + o\left(\frac{1}{n}\right) - (k - \frac{1}{2})\right],
\]

\[
= \frac{(n - \frac{1}{2})(1 - \delta)n + (2\delta - 1)k - \frac{1}{2}\delta}{2(n - \delta)} + \frac{(n - \frac{1}{2})^2\gamma}{(n - \delta)^2} + o(1),
\]

do not exist as \( n \to \infty \). That is (iii) does not exist.
If (iii) exists (say, the limit is $\sigma$), then, for all $k = 1, 2, \ldots, n - 1$ and $n$ sufficiently large, we have
\[ X^{(n)}_k = \frac{k - \frac{1}{2}}{n - \frac{1}{2}} + \frac{\sigma}{(n - \frac{1}{2})^2} + o\left(\frac{1}{n^2}\right). \]
Then, for all $k = 1, 2, \ldots, n - 1$,
\[
n[nX^{(n)}_k - k] = n\left[-\frac{n-k}{2n-1} + \frac{n\sigma}{(n-\frac{1}{2})^2} + o\left(\frac{1}{n}\right)\right],
\]
\[
= -\frac{n(n-k)}{2n-1} + \frac{n^2\sigma}{(n-\frac{1}{2})^2} + o(1),
\]
and
\[
(n-1)[(n-1)X^{(n)}_k - (k - \frac{1}{2})] = (n-1)\left[-\frac{k - \frac{1}{2}}{2n-1} + \frac{(n-1)\sigma}{(n-\frac{1}{2})^2} + o\left(\frac{1}{n}\right)\right],
\]
\[
= -\frac{(n-1)(k - \frac{1}{2})}{2n-1} + \frac{(n-1)^2\sigma}{(n-\frac{1}{2})^2} + o(1)
\]
do not exist as $n \to \infty$. That is, (i) and (ii) do not exist.

The cases (iv)–(vi) are similar. \hfill \Box

**Definition 2.2.** We say $X = \{X^{(n)}_k\} \in \Sigma$ satisfies condition A if one of (i)–(iii) holds and condition B if one of (iv)–(vi) holds.

(A) Let $k$ be fixed. We define $\alpha \in [0, \pi)$ such that $\alpha = 0$ if (i) exists, and
\[
-\cot \alpha = \begin{cases} 
\lim_{n \to \infty} (n-1)\pi^2[(n-1)X^{(n)}_k - (k - \frac{1}{2})] & \text{if (ii) exists}, \\
\lim_{n \to \infty} (n-\frac{1}{2})\pi^2[(n-\frac{1}{2})X^{(n)}_k - (k - \frac{1}{2})] & \text{if (iii) exists}.
\end{cases}
\]

(B) Let $k$ be fixed. We define $\beta \in [0, \pi)$ such that $\beta = 0$ if (iv) exists, and
\[
-\cot \beta = \begin{cases} 
\lim_{n \to \infty} (n-1)\pi^2[(n-1)X^{(n)}_{n-k} - (n-k - \frac{1}{2})] & \text{if (v) exists}, \\
\lim_{n \to \infty} (n-\frac{1}{2})\pi^2[(n-\frac{1}{2})X^{(n)}_k - (n-k)] & \text{if (vi) exists}.
\end{cases}
\]

**Definition 2.3.** We say $X$ is quasi-nodal to $(q, \alpha, \beta) \in \Omega$ if $X$ is an admissible sequence, satisfying conditions A and B, and $X$ also satisfies the following properties for each distinct case of $(\alpha, \beta)$.

**Case I** ($\alpha = \beta = 0$). $X$ has the following asymptotics uniformly for $k$ as $n \to \infty$:
\[ X^{(n)}_k = \frac{k}{n} + O\left(\frac{1}{n^2}\right), \]
and the sequence
\[ F_n(x) = 2n^2\pi^2[nL^{(n)}_x(x) - 1] \]
converges to $q$ in $L^1$. 

CASE II \((\alpha \beta > 0)\). \(X\) has the following asymptotics uniformly for \(k \rightarrow \infty\):

\[
X_k^{(n)} = \frac{k - \frac{1}{2}}{n} + O\left(\frac{1}{n^2}\right),
\]

and the sequence

\[
F_n(x) = 2(n - 1)^2 \pi^2 [(n - 1)L_{J_n(x)}^{(n)} - 1] + 2(\cot \beta - \cot \alpha)
\]

converges to \(q\) in \(L^1\).

CASE III \((\alpha = 0 < \beta)\). \(X\) has the following asymptotics uniformly for \(k \rightarrow \infty\):

\[
X_k^{(n)} = \frac{k - \frac{1}{2}}{n} + O\left(\frac{1}{n^2}\right),
\]

and the sequence

\[
F_n(x) = 2(n - \frac{1}{2})^2 \pi^2 [(n - \frac{1}{2})L_{J_n(x)}^{(n)} - 1] + 2 \cot \beta
\]

converges to \(q\) in \(L^1\).

CASE IV \((\beta = 0 < \alpha)\). \(X\) has the following asymptotics uniformly for \(k \rightarrow \infty\):

\[
X_k^{(n)} = \frac{k - \frac{1}{2}}{n} + O\left(\frac{1}{n^2}\right),
\]

and the sequence

\[
F_n(x) = 2(n - \frac{1}{2})^2 \pi^2 [(n - \frac{1}{2})L_{J_n(x)}^{(n)} - 1] - 2 \cot \alpha
\]

converges to \(q\) in \(L^1\).

In the cases above, we define \(X_0^{(n)} = 0\) and \(X_n^{(n)} = 1\).

\[
L_k^{(n)} = X_{k+1}^{(n)} - X_k^{(n)} \quad \text{for } 0 \leq k \leq n - 1
\]

is called a quasi-nodal length, and \(J_n(x) = \max\{k : X_k^{(n)} \leq x\}\). Define

\[
\Gamma = \{X = \{X_k^{(n)}\} : X \text{ is quasi-nodal to some } (q, \alpha, \beta) \in \Omega\}. \tag{2.1}
\]

**Lemma 2.4.** Suppose that \(X\) is quasi-nodal to some \((q, \alpha, \beta) \in L^1(0, 1) \times [0, \pi]^2\).

For all \(x\), let \(J_n(x) = \max\{k : X_k^{(n)} \leq x\}\) and \(a_n = n\) for all \(n\) or \(n - \frac{1}{2}\) for all \(n\). Then, for all \(g \in L^1(0, 1)\), the function

\[
G_n(x) = a_n \int_{J_n(x)}^{X_{J_n(x)+1}^{(n)}} (1 + \alpha_0 \cos(2a_n \pi t)) g(t) \, dt
\]

converges to \(g\) in \(L^1(0, 1)\), where \(\alpha_0 = 1\) or \(-1\).

**Proof.** The result follows from [2, lemma 2.3], [9, theorem 3.2], and the fact that

\[
\lim_{n \to \infty} \frac{s_n}{a_n \pi} = 1 \quad \text{and} \quad L_k^{(n)} = \frac{1}{n} + O\left(\frac{1}{n^2}\right).
\]
Theorem 2.5. Let \( \{X_k^{(n)}\} \) be quasi-nodal to some \( (q, \alpha, \beta) \in L^1(0,1) \times [0, \pi)^2 \). Then we have the asymptotics of \( L_k^{(n)} \) for different cases as the following:

(I) for all \( k = 1, 2, \ldots, n - 2 \),

\[
L_0^{(n)} = \frac{1}{n} + o\left(\frac{1}{n^2}\right), \\
L_{n-1}^{(n)} = \frac{1}{n} + o\left(\frac{1}{n^2}\right), \\
L_k^{(n)} = \frac{1}{n} + o\left(\frac{1}{n^2}\right);
\]

(II) for all \( k = 1, 2, \ldots, n - 2 \)

\[
L_0^{(n)} = \frac{1}{2(n-1)} + o\left(\frac{1}{n^2}\right), \\
L_{n-1}^{(n)} = \frac{1}{2(n-1)} + o\left(\frac{1}{n^2}\right), \\
L_k^{(n)} = \frac{1}{n-1} + o\left(\frac{1}{n^2}\right);
\]

(III) for all \( k = 1, 2, \ldots, n - 2 \)

\[
L_0^{(n)} = \frac{1}{n-\frac{1}{2}} + o\left(\frac{1}{n^2}\right), \\
L_{n-1}^{(n)} = \frac{1}{2n-1} + o\left(\frac{1}{n^2}\right), \\
L_k^{(n)} = \frac{1}{n-\frac{1}{2}} + o\left(\frac{1}{n^2}\right);
\]

(IV) for all \( k = 1, 2, \ldots, n - 2 \)

\[
L_0^{(n)} = \frac{1}{2n-1} + o\left(\frac{1}{n^2}\right), \\
L_{n-1}^{(n)} = \frac{1}{n-\frac{1}{2}} + o\left(\frac{1}{n^2}\right), \\
L_k^{(n)} = \frac{1}{n-\frac{1}{2}} + o\left(\frac{1}{n^2}\right);
\]

Proof. Suppose \( X \) belongs to case II. Then, by the definition of \( X \) and lemma 2.4,

\[
2(n-1)^2 \pi^2 [(n-1)L_{J_{n-1}}^{(n)} - 1] + 2(cot \beta - cot \alpha) - (n-1) \int_{X_{J_{n}^{(n)}}}^{X_{J_{n-1}}^{(n)}} (1 - \cos(2(n-1)\pi t))q(t) \, dt
\]
converges to zero in $L^1(0,1)$. Thus, as $n \to \infty$,

$$\int_0^1 \left| L_{J_n(x)}^{(n)} \right| = \left| \left[ \frac{1}{n-1} + \frac{1}{2(n-1)^2 \pi^2} \right. \right.$$  

$$\times \int_{X_{J_n(x)}^{(n)}}^{X_{J_n(x)+1}^{(n)}} (1 - \cos(2(n-1)\pi t)) q(t) \, dt + O\left( \frac{1}{n^3} \right) \left. \right| \, dx = o\left( \frac{1}{n^3} \right).$$

Hence,

$$\sum_{k=0}^{n-1} L_{k}^{(n)} L_{k}^{(n)} = \left[ \frac{1}{n-1} + \frac{1}{2(n-1)^2 \pi^2} \right. \right.$$  

$$\times \int_{X_{k}^{(n)}}^{X_{k+1}^{(n)}} (1 - \cos(2(n-1)\pi t)) q(t) \, dt + O\left( \frac{1}{n^3} \right) \right| = o\left( \frac{1}{n^3} \right).$$

Since

$$L_{0}^{(n)} = \frac{1}{2(n-1)} + O\left( \frac{1}{n^2} \right), \quad L_{n-1}^{(n)} = \frac{1}{2(n-1)} + O\left( \frac{1}{n^2} \right), \quad L_{k}^{(n)} = \frac{1}{n-1} + O\left( \frac{1}{n^2} \right)$$

for all $k = 1, 2, \ldots, n-2$, we obtain

$$L_{k}^{(n)} = \frac{1}{n-1} + \frac{1}{2(n-1)^2 \pi^2} \int_{X_{k}^{(n)}}^{X_{k+1}^{(n)}} (1 - \cos(2(n-1)\pi t)) q(t) \, dt$$  

$$+ O\left( \frac{1}{n^3} \right) + o\left( \frac{1}{n^2} \right),$$

$$= \frac{1}{n-1} + o\left( \frac{1}{n^2} \right).$$

Similarly

$$L_{0}^{(n)} = \frac{1}{2(n-1)} + o\left( \frac{1}{n^2} \right), \quad L_{n-1}^{(n)} = \frac{1}{2(n-1)} + o\left( \frac{1}{n^2} \right).$$

\[ \square \]

3. Proof of the main theorem

**Definition 3.1.** Let $X, \bar{X} \in \Gamma$, define

$$S_n(X, \bar{X}) \equiv n^2 \pi^2 \sum_{k=0}^{n-1} |L_{k}^{(n)} - L_{k}^{(n)}|,$$

$$d_0(X, \bar{X}) \equiv \lim_{n \to \infty} S_n(X, \bar{X}),$$

$$d_{\Gamma}(X, \bar{X}) \equiv \lim_{n \to \infty} \frac{S_n(X, \bar{X})}{1 + S_n(X, \bar{X})}.$$

**Lemma 3.2.** $d_{\Gamma}(\cdot, \cdot)$ is a pseudometric on $\Gamma$. 
Remark 3.3. Note that our definition of $d_\Gamma$ is the same as that of $d_\Sigma$ in [8]. Thus, $d_\Gamma(X, \bar{X}) \leq d_0(X, \bar{X})$. If $d_0(X, \bar{X}) < \infty$, then

$$d_0(X, \bar{X}) \leq \frac{d_\Gamma(X, \bar{X})}{1 - d_\Gamma(X, \bar{X})}.$$ 

This means that $d_0(X, \bar{X})$ is close to zero if and only if $d_\Gamma(X, \bar{X})$ is close to zero. In particular, $d_0(X, \bar{X}) = 0$ if and only if $d_\Gamma(X, \bar{X}) = 0$.

Lemma 3.4. Suppose $X, \bar{X} \in \Gamma$ belong to different cases. Then

$$d_\Gamma(X, \bar{X}) = 1.$$

Proof. Suppose $X$ belongs to case I and $\bar{X}$ belongs to case II. Then

$$L_0^{(n)} - \bar{L}_0^{(n)} = \frac{n - 2}{2n(n - 1)} + o\left(\frac{1}{n^2}\right),$$

$$L_{n-1}^{(n)} - \bar{L}_{n-1}^{(n)} = \frac{n - 2}{2n(n - 1)} + o\left(\frac{1}{n^2}\right),$$

$$L_k^{(n)} - \bar{L}_k^{(n)} = -\frac{1}{n(n - 1)} + o\left(\frac{1}{n^2}\right), \quad k = 1, 2, \ldots, n - 2.$$

Hence,

$$d_\Gamma(X, \bar{X}) = \lim_{n \to \infty} \frac{n^2 \pi^2 \sum_{k=0}^{n-1} |L_k^{(n)} - \bar{L}_k^{(n)}|}{1 + n^2 \pi^2 \sum_{k=0}^{n-1} |L_k^{(n)} - \bar{L}_k^{(n)}|} = \frac{2n(n - 2)\pi^2(n - 1)^{-1} + o(n)}{1 + 2n(n - 2)\pi^2(n - 1)^{-1} + o(n)} = 1.$$

Suppose $X$ belongs to case I and $\bar{X}$ belongs to case IV. Then

$$d_\Gamma(X, \bar{X}) = \lim_{n \to \infty} \frac{n(2n - 2)\pi^2(2n - 1)^{-1} + o(1) + o(n)}{1 + n(2n - 2)\pi^2(2n - 1)^{-1} + o(1) + o(n)} = 1.$$

Lemma 3.5. Suppose $X, \bar{X} \in \Gamma$ belong to the same case. Then, when $n$ is sufficiently large,

(i) the interval $I_k^{(n)}$ between $\{X_k^{(n)}\}$ and $\{\bar{X}_k^{(n)}\}$ has length $O\left(\frac{1}{n^2}\right)$;

(ii) for all $x \in (0, 1)$, $|J_n(x) - \bar{J}_n(x)| \leq 1$.

The proof is similar to [8, lemma 3.3] and is omitted.

Proof of theorem 1.4. It suffices to show that if $X, \bar{X}$ are quasi-nodal to $(\bar{q}, \bar{\alpha}, \bar{\beta})$ and $(\tilde{q}, \tilde{\alpha}, \tilde{\beta})$ in $\Omega$, respectively, then
On the quasi-nodal map for the Sturm–Liouville problem

(a) \[\|q - \bar{q}\|_1 + |\cot \alpha - \cot \bar{\alpha}| + |\cot \beta - \cot \bar{\beta}| \leq 8d_0(X, \bar{X}),\]

(b) \[2d_0(X, \bar{X}) \leq \|q - \bar{q}\|_1 + 2|\cot \alpha - \cot \bar{\alpha}| + 2|\cot \beta - \cot \bar{\beta}|,\]

when \((q, \alpha, \beta)\) and \((\bar{q}, \alpha, \bar{\beta})\) belong to the same case.

The proof of part (a) is exactly the same as that of [8, theorem 3.4]. Consider case I in part (b). Since \(2n^2 \pi^2 (nL_{jn(x)}^{(n)} - 1)\) converges to \(q\) in \(L^1(0,1)\) and \(2n^2 \pi^2 (n\bar{L}_{jn(x)}^{(n)} - 1)\) converges to \(\bar{q}\) in \(L^1(0,1)\), when \(n\) is sufficiently large, we have

\[
\|q - \bar{q}\|_1 + o(1) = \int_0^1 2n^3 \pi^2 |L_{jn(x)}^{(n)} - \bar{L}_{jn(x)}^{(n)}| \, dx,
\]

\[
\geq \int_0^1 2n^3 \pi^2 |L_{jn(x)}^{(n)} - \bar{L}_{jn(x)}^{(n)}| \, dx - \int_0^1 2n^3 \pi^2 |\bar{L}_{jn(x)}^{(n)} - \bar{L}_{jn(x)}^{(n)}| \, dx,
\]

where the second term on the right-hand side is \(o(1)\). Hence, as \(n \to \infty\),

\[
2n^2 \pi^2 \sum_{k=0}^{n-1} |L_k^{(n)} - \bar{L}_k^{(n)}| \leq \|q - \bar{q}\|_1 + o(1).
\]

That is

\[
2d_0(X, \bar{X}) \leq \|q - \bar{q}\|_1.
\]

For case II, by the convergence of \(F_n\) and \(\bar{F}_n\),

\[
\|q - \bar{q}\|_1 + o(1)
\]

\[
= \|2(n - 1)^2 \pi^2 [(n - 1)L_{jn(x)}^{(n)} - 1] + 2(\cot \beta - \cot \alpha)
\]

\[- 2(n - 1)^2 \pi^2 [(n - 1)\bar{L}_{jn(x)}^{(n)} - 1] - 2(\cot \beta - \cot \bar{\alpha})\|_1,
\]

\[
\geq \|2(n - 1)^3 \pi^2 (L_{jn(x)}^{(n)} - \bar{L}_{jn(x)}^{(n)})\|_1 - 2|\cot \alpha - \cot \bar{\alpha}| - 2|\cot \beta - \cot \bar{\beta}|.
\]

Hence,

\[
\|2n^3 \pi^2 (L_{jn(x)}^{(n)} - \bar{L}_{jn(x)}^{(n)})\|_1 = \frac{n^3}{(n - 1)^3} \|2(n - 1)^3 \pi^2 (L_{jn(x)}^{(n)} - \bar{L}_{jn(x)}^{(n)})\|_1
\]

\[
\leq \|q - \bar{q}\|_1 + 2|\cot \alpha - \cot \bar{\alpha}| + 2|\cot \beta - \cot \bar{\beta}| + o(1).
\]

By a similar argument as in case I, we obtain

\[
2d_0(X, \bar{X}) \leq \|q - \bar{q}\|_1 + 2|\cot \alpha - \cot \bar{\alpha}| + 2|\cot \beta - \cot \bar{\beta}|.
\]

4. Two examples

In this section, we give two examples to illustrate that it is \(L^1\) convergence that matters in the definition of quasi-nodal sets. Consider the simplest Sturm–Liouville problem:

\[
- y'' = \lambda y,
\]

\[y(0) = y(1) = 0.
\]
Then the nodal set is
\[ X = \left\{ x^{(n)}_k = \frac{k}{n} : k = 1, 2, \ldots, n-1; \ n > 1 \right\}, \]
and the nodal length is
\[ l^{(n)}_k = \frac{1}{n}, \quad \text{for all } k = 0, 1, \ldots, n-1 \text{ and } n \in \mathbb{N}. \]
For all \( x \in (0, 1) \), define \( F_n(x) = 2n^2 \pi^2 [nL^{(n)}_k(x) - 1] \). Note that if \( L^{(n)}_k = l^{(n)}_k \) for all \( k = 0, 1, \ldots, n-1 \) and \( n \in \mathbb{N} \), then \( F_n(x) = 0 \) for all \( x \in (0, 1) \) and \( n \in \mathbb{N} \).

### 4.1. Example 1

Let
\[ \bar{X}^{(n)}_k = \begin{cases} \frac{k}{n} + \frac{1}{2n^2 \pi^2}, & k = \lfloor \frac{1}{2} n \rfloor, \\ \frac{k}{n}, & \text{otherwise}. \end{cases} \]
Then
\[ \bar{L}^{(n)}_k = \begin{cases} \frac{1}{n} + \frac{1}{2n^2 \pi^2}, & k = \lfloor \frac{1}{2} n \rfloor, \\ \frac{1}{n} - \frac{1}{2n^2 \pi^2}, & k = \lfloor \frac{1}{2} n \rfloor + 1, \\ \frac{1}{n}, & \text{otherwise}. \end{cases} \]
Hence,
\[ d_{\Gamma}(X, \bar{X}) = \lim_{n \to \infty} \frac{S_n(X, \bar{X})}{1 + S_n(X, X)} = \lim_{n \to \infty} \frac{1}{1 + 1} = \frac{1}{2}, \]
and
\[ \int_0^1 |\bar{F}_n(x)| \, dx = \sum_{k=0}^{n-1} 2n^2 \pi^2 [n\bar{L}^{(n)}_k - 1] \to 2. \]
Moreover, for all \( x \in (0, 1) \) \( \setminus \{ \frac{1}{2} \} \), let \( N > 2/|x - \frac{1}{2}| \). Then for all \( n > N, x \notin \left( \bar{X}^{(n)}_{\lfloor n/2 \rfloor - 1}, \bar{X}^{(n)}_{\lfloor n/2 \rfloor + 1} \right) \), \( F_n(x) \to 0 \) as \( n \to \infty \). That is, \( F_n(x) \to 0 \) a.e., but not in \( L^1(0, 1) \), and \( d_{\Gamma}(X, \bar{X}) \neq 0 \).

### 4.2. Example 2

Define
\[ X^{(2)}_1 = \frac{1}{2} + \frac{1}{16\pi^2}, \quad X^{(3)}_1 = \frac{1}{3} + \frac{1}{2 \cdot 3^2 \pi^2}, \quad \bar{X}^{(3)}_2 = \frac{2}{3}, \]
\[ X^{(4)}_1 = \frac{1}{4}, \quad \bar{X}^{(4)}_2 = \frac{1}{2}, \quad \bar{X}^{(4)}_3 = \frac{3}{4} + \frac{1}{2 \cdot 4^3 \pi^2}. \]
Thus,

\[
F_2(x) = \begin{cases} 
1, & x \leq \frac{1}{2}, \\
-1, & x > \frac{1}{2},
\end{cases}
\]

\[
F_3(x) = \begin{cases} 
1, & x \leq \frac{1}{3} + \frac{1}{2 \cdot 3^3 \pi^2}, \\
-1, & \frac{1}{3} + \frac{1}{2 \cdot 3^3 \pi^2} < x \leq \frac{2}{3}, \\
0, & x > \frac{2}{3},
\end{cases}
\]

\[
F_4(x) = \begin{cases} 
0, & 0 \leq x \leq \frac{1}{2}, \\
1, & \frac{1}{2} < x \leq \frac{3}{4} + \frac{1}{2 \cdot 4^3 \pi^2}, \\
-1, & x > \frac{3}{4} + \frac{1}{2 \cdot 4^3 \pi^2}.
\end{cases}
\]

Inductively, when \( \frac{1}{2^m(m+1)} < n - 1 < \frac{1}{2(m+1)(m+2)} \), let \( r = n - 1 - \frac{1}{2^m(m+1)} \).

Then define

\[
X_{k}^{(n)} = \begin{cases} 
\frac{k}{n}, & k \notin \left[ \frac{r}{m}, \frac{r+1}{m} \right] \text{ or } k \text{ is even}, \\
\frac{k}{n} + \frac{1}{2n^3\pi^2}, & \frac{k}{n} \in \left[ \frac{r}{m}, \frac{r+1}{m} \right] \text{ and } k \text{ is odd}.
\end{cases}
\]

Hence,

\[
L_{k}^{(n)} = \begin{cases} 
\frac{1}{n} + \frac{(-1)^{k+1}}{2n^3\pi^2}, & \frac{rn}{m} \leq k \leq \left\lfloor \frac{(r+1)n}{m} \right\rfloor + 1, \\
\frac{1}{n}, & \text{otherwise}.
\end{cases}
\]

Therefore, with \( J = J_n(x) \),

\[
F_n(x) = 2n^2\pi^2|nL_k^{(n)} - 1| = \begin{cases} 
(-1)^{r+1}, & \frac{rn}{m} \leq nx < \left\lfloor \frac{(r+1)n}{m} \right\rfloor + 1, \\
0, & \text{otherwise}.
\end{cases}
\]

In this way, \( F_n(x) \) is not convergent pointwisely, but

\[
\int_0^1 |F_n(x)| \, dx = \sum_{k=0}^{n-1} L_k^{(n)}|2n^2\pi^2|nL_k^{(n)} - 1|, 
\]

\[
= \left( \frac{1}{n} + o \left( \frac{1}{n^2} \right) \right) m.
\]
and so converges to zero as $n \to \infty$. Furthermore,
\[ S_n(X, \bar{X}) = \frac{m}{2n}, \]
and so converges to zero as $n \to \infty$. As a result,
\[ d_F(X, \bar{X}) = \lim_{n \to \infty} \frac{S_n(X, \bar{X})}{1 + S_n(X, \bar{X})} = 0. \]

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Appendix A.
Let $y_1(x, \lambda), y_2(x, \lambda)$ be solutions of $-y'' + q(x)y = \lambda y$, such that they satisfy the initial conditions
\[ y_1(0) = y_2(0) = 1, \]
\[ y'_1(0) = y_2(0) = 0, \]
where $q \in L^1(0, 1)$ and $\lambda \in \mathbb{C}$. The idea of the following lemma comes from [12, lemma 1.7], which can be extended to the case $q \in L^1(0, 1)$.

**LEMMA A.1.** Let $|\sqrt{\lambda}| > 2\|q\|_1$. We then have the following estimates:
\[ |y_1(x, \lambda)| < 2e^{\|q\|_1|\sqrt{\lambda}|x}, \]
\[ |y_2(x, \lambda)| < 2\frac{e^{\|q\|_1|\sqrt{\lambda}|x}}{|\sqrt{\lambda}|}, \]
\[ |y_1(x, \lambda) - \cos(\sqrt{\lambda}x)| < \frac{2\|q\|_1e^{\|q\|_1|\sqrt{\lambda}|x}}{|\sqrt{\lambda}|}, \]
\[ |y_2(x, \lambda) - \sin(\sqrt{\lambda}x)| < \frac{2\|q\|_1e^{\|q\|_1|\sqrt{\lambda}|x}}{|\lambda|}, \]
and
\[ |y'_1(x, \lambda) + \sqrt{\lambda}\sin(\sqrt{\lambda}x)| < \frac{2\|q\|_1e^{\|q\|_1|\sqrt{\lambda}|x}}{|\sqrt{\lambda}|}, \]
\[ |y'_2(x, \lambda) - \cos(\sqrt{\lambda}x)| < \frac{2\|q\|_1e^{\|q\|_1|\sqrt{\lambda}|x}}{|\sqrt{\lambda}|}. \]

Suppose $\phi(x, \lambda)$ is a solution of the following initial-value problem:
\[ -y'' + q(x)y = \lambda y, \]
\[ y(0) = \sin \alpha, \]
\[ y'(0) = -\cos \alpha. \]
Then \( \phi \) can be represented as a linear combination of \( y_1 \) and \( y_2 \):
\[
\phi(x, \lambda) = y_1(x, \lambda) \sin \alpha - y_2(x, \lambda) \cos \alpha.
\]
Moreover, on letting \( F(\lambda) = \phi(1, \lambda) \cos \beta + \phi'(1, \lambda) \sin \beta \), \( \mu \) is an eigenvalue if and only if \( F(\mu) = 0 \). In the following, we will show the counting lemma for the case \( \alpha, \beta > 0 \) which was confusing in some papers [8,9,13].

**Lemma A.2** (Poschel and Trubowitz [11, lemma 2.1]). If \(|z - n\pi| \geq \frac{1}{2} \pi\), then
\[
e^{\text{Im} z} < 4|\sin z|.
\]

**Theorem A.3.** Suppose \( \alpha, \beta > 0 \) and let \( N > 16(||q||_1 + 1)(|\cot \alpha| + |\cot \beta| + 1) + 8|\cot \alpha \cot \beta|^{1/2} \) be an integer. Then \( F(\lambda) \) has exactly \( N + 1 \) roots, counted with multiplicities, in the open half-plane
\[
\Re \lambda < (N + \frac{1}{2})^2 \pi^2,
\]
and, for each \( n > N \), exactly one simple root in the egg-shaped region
\[
|\sqrt{\lambda} - n\pi| < \frac{1}{2} \pi.
\]
There are no other roots.

**Proof.** Consider the contours
\[
|\sqrt{\lambda}| = (N + \frac{3}{2}) \pi, \quad \Re \sqrt{\lambda} = (N + \frac{1}{2}) \pi \quad \text{and} \quad |\sqrt{\lambda} - n\pi| = \frac{1}{2} \pi \quad \text{for} \ n > N.
\]
On these contours, \( |\sqrt{\lambda}| > 2N, \ |\sqrt{\lambda} - n\pi| > \frac{1}{2} \pi \) and, by lemmas A.1 and A.2,
\[
\left| \frac{F(\lambda)}{\sin \alpha \sin \beta} - \sqrt{\lambda} \sin(\sqrt{\lambda}) \right| \\
\leq \left| y_1'(1, \lambda) + \sqrt{\lambda} \sin(\sqrt{\lambda}) \right| + \left| y_2'(1, \lambda) \cot \alpha \right| \\
+ \left| y_1(1, \lambda) \right| |\cot \beta| + \left| y_2(1, \lambda) \right| \cot \alpha \cot \beta \\
\leq e^{\text{Im}(\sqrt{\lambda})} \left( 2 ||q||_1 + 2 \frac{|\cot \alpha|}{|\sqrt{\lambda}|} + 2 |\cot \beta| + 2 |\cot \alpha \cot \beta| \right) \\
< |\sqrt{\lambda} \sin(\sqrt{\lambda})|.
\]
Since \( \sin(\sqrt{\lambda}) \) is an entire function, it can be rewritten as a Taylor series and hence
\[
\sqrt{\lambda} \sin(\sqrt{\lambda}) = \sqrt{\lambda} \sum_{i=0}^{\infty} \frac{(-1)^i (\sqrt{\lambda})^{2i+1}}{(2i+1)!} = \lambda \sum_{i=0}^{\infty} \frac{\lambda^i}{(2i+1)!}.
\]
This implies that \( \lambda = 0 \) is a simple root of \( \sqrt{\lambda} \sin(\sqrt{\lambda}) \).

Since \( \sqrt{\lambda} \sin(\sqrt{\lambda}) \) has \( N + 1 \) roots in the domain bounded by \( |\sqrt{\lambda}| = (N + \frac{3}{2}) \pi \) and \( \Re \sqrt{\lambda} = (N + \frac{1}{2}) \pi \), and has a simple root in the domain \( |\sqrt{\lambda} - n\pi| < \frac{1}{2} \pi \), for all \( n > N \), by Rouche’s theorem, so does \( F(\lambda) \). This implies that the \( n \)th eigenvalue, \( n > N \), satisfies \( |\sqrt{\lambda} - (n-1)\pi| < \frac{1}{2} \pi \). \( \square \)
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References


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