12.6 Polar Coordinates

Consider Poisson’s equation on the unit disk.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = f(r, \theta)
\]

with \(0 \leq r \leq 1\), and \(0 \leq \theta \leq 2\pi\).

We approximate the equation by

\[
\frac{1}{r_i} \left( r_i + \frac{1}{2} \frac{u_{i+1,j} - u_{ij}}{\Delta r} - r_i - \frac{1}{2} \frac{u_{ij} - u_{i-1,j}}{\Delta r} \right) \frac{1}{\Delta r}
\]

\[
+ \frac{1}{r_i^2} \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{\Delta \theta^2} = f_{ij}
\]

where \(u_{ij}\) and \(f_{ij}\) are the grid functions at \((r_i, \theta_j) = (i \Delta r, j \Delta \theta)\).

The grid functions are periodic in \(j\) with period \(J = \frac{2\pi}{\Delta \theta}\), and \(u_{0j}\) is independent of the value of \(j\).

The main new feature of polar coordinates is the condition that must be imposed at the origin. (Since \(r = 0\) at origin)

Rmk: It’s important to realize that any difficulties that arise at the origin are only a result of the choice of coordinate system and are not reflected in the continuous function \(u(r, \theta)\).
Let $D = Disc(0, \varepsilon)$

\[
\int \int_{D} frdr \, d\theta = \int \int_{D} \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \right] + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} r dr \, d\theta
\]

\[
= \int_{0}^{2\pi} \frac{\partial u}{\partial r} \varepsilon d\theta
\]

Now choose $\varepsilon = \frac{\Delta r}{2}$ and approximate it by

\[
f(0)(\frac{\Delta r}{2})^2 \pi = \sum_{j=1}^{J} \frac{u_{1j} - u_{0} \Delta r}{\Delta r} \frac{\Delta r}{2} \Delta \theta.
\]

i.e. $u_{0} = \frac{1}{J} \sum_{j=1}^{J} u_{1j} - f(0)(\frac{\Delta r}{2})^2, \quad \Delta \theta = \frac{2\pi}{J}$


12.7 Coordinates Changes and Finite Differences

Frequently we must solve a system of partial differential equations on a domain that is not a rectangle, disk, or other nice shape. It is then desirable to change coordinates so that a convenient coordinate system can be used. To illustrate the techniques and the difficulties, we will work through a relatively simple example. It is not hard to come up with much more difficult examples.

We consider Poisson’s equation on the trapezoidal domain given by $0 \leq x \leq 1$ and $0 \leq y \leq \frac{1+x}{2}$. We take the new coordinate system

$$
\xi = x, \quad \eta = \frac{2y}{1 + x}
$$

so that $(\xi, \eta)$ in the unit square maps one-to-one with $(x, y)$ in the trapezoid.

To change coordinates we use the differentiation formulas

$$
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{2y}{(1 + x)^2} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\eta}{1 + \xi} \frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial y} = \frac{2}{1 + x} \frac{\partial}{\partial \eta} = \frac{2}{1 + \xi} \frac{\partial}{\partial \eta}
$$
Using these relations, Poisson’s equation becomes

\[ u_{\xi\xi} - \frac{\eta}{1 + \xi} u_{\xi\eta} - \left( \frac{\eta}{1 + \xi} u_{\eta} \right)_{\xi} + \frac{\eta}{(1 + \xi)^2} (\eta u_{\eta})_{\eta} \]

\[ + \frac{4}{(1 + \xi)^2} u_{\eta\eta} = f(\xi, \eta) \quad (12.7.1) \]

If we were to discretize (12.7.1) in this form using second-order accurate central differences, the matrix arising from the matrix representation would not be symmetric. Since the iterative solution methods we will study in the next two chapters will work if the matrix symmetric, we will show how to modify the equation (12.7.1) to obtain a symmetric, positive definite matrix. To do this we must get the equation in divergence form, that is, in the form

\[ \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) = \tilde{f} \]

where \( (x_1, x_2) = (\xi, \eta) \) and \( a_{ij} \) is a symmetric matrix at each point \( (x_1, x_2) \). If we multiply (12.7.1) by \( (1+\xi) \), we can collect terms such that (12.7.1) is equivalent to

\[ [(1+\xi)u_{\xi}]_{\xi} - (\eta u_{\xi})_{\eta} - (\eta u_{\eta})_{\xi} + \left( \frac{4 + \eta^2}{1 + \xi} u_{\eta} \right)_{\eta} = (1+\xi)f \]
This equation may be discretized on a uniform grid in $\xi$ and $\eta$ as
\[
\frac{(1 + \xi_{i+\frac{1}{2}})(u_{i+1,j} - u_{ij}) - (1 + \xi_{i-\frac{1}{2}})(u_{ij} - u_{i-1,j})}{\Delta \xi^2}
\]
\[
- \frac{\eta_{j+1}(u_{i+1,j+1} - u_{i-1,j+1}) - \eta_{j-1}(u_{i+1,j-1} - u_{i-1,j-1})}{4\Delta \xi \Delta \eta}
\]
\[
- \frac{\eta_j(u_{i+1,j+1} - u_{i,j}) - \eta_j(u_{i-1,j+1} - u_{i-1,j-1})}{4\Delta \xi \Delta \eta}
\]
\[
+ \frac{(4 + \eta_{j+\frac{1}{2}}^2)(u_{ij+1} - u_{ij}) - (4 + \eta_{j-\frac{1}{2}}^2)(u_{ij} - u_{ij-1})}{(1 + \xi_i)\Delta \eta^2}
\]
\[
= (1 + \xi_i)f_{ij} \tag{12.7.2}
\]

The matrix corresponding to this discretization is symmetric.

To show that the matrix is symmetric, we notice that the coefficient of $u_{i+1,j+1}$ in the equation at $(i, j)$ is \[-\frac{\eta_{j+1} + \eta_j}{4\Delta \xi \Delta \eta},\] and this is also the coefficient of $u_{ij}$ in the equation at $(i + 1, j + 1)$. The same is true for all the other nonzero coefficients. Thus the matrix is symmetric.

To show that the matrix of the equations in (12.7.2) is negative definite, we consider the operator on the left-hand side of (12.7.2) applied to a grid function $\phi_{ij}$ that is zero
on the boundaries of the unit \((\xi, \eta)\) square. Multiplying
the operator applied to \(\phi\) at \((i, j)\) by \(\phi_{ij}\) and summing
over all \((i, j)\) gives a long expression that we will consider
in three parts. Denote these sums by \(\sum_{\xi}\), \(\sum_{\eta}\), and
\(\sum_{\xi\eta}\). The terms from the second difference in \(\xi\) are
\[
\sum_{\xi\xi} = \Delta \xi^{-2} \sum \phi_{ij} [(1 + \xi_{i + \frac{1}{2}})(\phi_{i+1j} - \phi_{ij}) - (1 + \xi_{i - \frac{1}{2}})(\phi_{ij} - \phi_{i-1j})]
\]
\[
= -\Delta \xi^{-2} \sum (1 + \xi_{i + \frac{1}{2}})(\phi_{i+1j} - \phi_{ij})^2
\]
by summation by parts. Similarly, the terms form the
second differences in \(\eta\) are
\[
\sum_{\eta\eta} = -\Delta \eta^{-2} \sum \frac{4 + \eta_{j + \frac{1}{2}}}{1 + \xi_i}(\phi_{ij+1} - \phi_{ij})^2
\]
The sums are over all interior \((i, j)\) values. The sums
from the mixed differences are also treated by summation
by parts and become
\[
\sum_{\xi\eta} = (2\Delta \xi \Delta \eta)^{-1} \sum \eta_j(\phi_{i+1j} - \phi_{i-1j})(\phi_{ij+1} - \phi_{ij-1})
\]
To show that the matrix is negative definite, we must show
that
\[- \left( \sum_{\xi\xi} + \sum_{\xi\eta} + \sum_{\eta\eta} \right) \geq -C \left( \sum_{\xi\xi} + \sum_{\eta\eta} \right) \geq 0\]
for some positive number \( c \). This is easily done, as follows

\[- \eta_j \frac{(\phi_{i+1,j} - \phi_{i-1,j})(\phi_{ij+1} - \phi_{ij-1})}{2\Delta \xi \Delta \eta} \leq a(1 + \xi_i) \left( \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta \xi} \right)^2 + \frac{1}{a 1 + \xi_i} \left( \frac{\phi_{ij+1} - \phi_{ij-1}}{2\Delta \eta} \right)^2 \]
\[\leq \frac{a}{2}(1 + \xi_i) \left[ \left( \frac{\phi_{i+1,j} - \phi_{ij}}{\Delta \xi} \right)^2 + \left( \frac{\phi_{ij} - \phi_{i-1,j}}{\Delta \xi} \right)^2 \right] \]
\[+ \frac{1}{2a 1 + \xi_i} \left[ \left( \frac{\phi_{ij+1} - \phi_{ij}}{\Delta \eta} \right)^2 + \left( \frac{\phi_{ij} - \phi_{ij-1}}{\Delta \eta} \right)^2 \right] \]

Therefore,
\[- \left( \sum_{\xi\xi} + \sum_{\xi\eta} + \sum_{\eta\eta} \right) \leq (1-a)\Delta \xi^{-2} \sum (1+\xi_{i+\frac{1}{2}})(\phi_{i+1,j} - \phi_{ij})^2 \]
\[+ \Delta \eta^{-2} \sum \frac{4-\eta_{j+\frac{1}{2}}^2}{1+\xi_i} \left( \phi_{ij+1} - \phi_{ij} \right)^2.\]

If we choose \( a = \frac{1}{2} \), say, then both sums are nonnegative. Thus the system of difference equation (12.7.2) has a matrix that is symmetric and negative definite.

This system of equations can be solved by the methods discussed in Chapters 13 and 14.