Coefficients of the characteristic polynomial

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Overivew of the characteristic polynomial

The characteristic polynomial of A is defined by $p_A(x) = \det(A - xI)$. Usually, we need to expand every term and rearrange it into a clean form. For example,

$$\begin{aligned}\det(A - xI) &= \det\begin{bmatrix} 1-x & 2 & 3 \\ 4 & 5-x & 6 \\ 7 & 8 & 9-x \end{bmatrix} \\ &= (1-x)(5-x)(9-x) + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8 \\ &\quad -3\cdot(5-x)\cdot 7 - 2\cdot 4\cdot(9-x) - (1-x)\cdot 6\cdot 8 \\ &= -x^3 + 15x^2 + 18x.\end{aligned}$$

The following formula helps to find the coefficients directly. The characteristic polynomial of A can be written as

$$p_A(x) = \det(A - xI) = (-x)^n + s_1(-x)^{n-1} + \cdots + s_n,$$

where

$$s_k = \sum_{\substack{\alpha\subseteq[n] \\ |\alpha|=k}} \det(A[\alpha]).$$

An example

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We have to go through all subsets α of $[3] = \{1,2,3\}$ of a given size.

	$k=0$	$k=1$			$k=2$			$k=3$
α	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$A[\alpha]$	$[\,]$	$[1]$	$[5]$	$[9]$	$\begin{bmatrix}1&2\\4&5\end{bmatrix}$	$\begin{bmatrix}1&3\\7&9\end{bmatrix}$	$\begin{bmatrix}5&6\\8&9\end{bmatrix}$	$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$
det	1	1	5	9	-3	-12	-3	0
	$s_0 = 1$	$s_1 = 15$			$s_2 = -18$			$s_3 = 0$

Thus,

$$\begin{aligned}p_A(x) &= (-x)^n + s_1(-x)^{n-1} + \cdots + s_n, \\ &= -x^3 + 15x^2 + 18x.\end{aligned}$$

Some theory behind the scene

When we compute the characteristic polynomial, each column of $A - xI$ can be written as the sum of two vectors, one contains constants while the other contains $-x$ and 0's. By expanding $\det(A - xI)$ using the distributive law on each column, we got 2^n terms.

$$\begin{aligned}&\det\begin{bmatrix} 1-x & 2 & 3 \\ 4 & 5-x & 6 \\ 7 & 8 & 9-x \end{bmatrix} \\ &= \det\begin{bmatrix} -x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & -x \end{bmatrix} + \det\begin{bmatrix} -x & 0 & 3 \\ 0 & -x & 6 \\ 0 & 0 & 9 \end{bmatrix} + \det\begin{bmatrix} -x & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 8 & -x \end{bmatrix} + \det\begin{bmatrix} 1 & 0 & 0 \\ 4 & -x & 0 \\ 7 & 0 & -x \end{bmatrix} + \\ &\det\begin{bmatrix} -x & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 8 & 9 \end{bmatrix} + \det\begin{bmatrix} 1 & 0 & 3 \\ 4 & -x & 6 \\ 7 & 0 & 9 \end{bmatrix} + \det\begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & -x \end{bmatrix} + \det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ &= (-x)^3 + (9+5+1)(-x)^2 + (-3-12-3)(-x) + 0.\end{aligned}$$

My own example

I wrote some SageMath code to compute the characteristic polynomial using this method. Visit the website below to see more examples.

https://sagecell.sagemath.org/?q=zziwzi