Applications of zero forcing number to the minimum rank problem

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Abstract

- Introduction and some related properties
- Exhaustive zero forcing number and sieving process
- Summary and a counterexample to a problem on edge spread
Relation between Matrices and Graphs

\[ G : \text{real symmetric matrices} \rightarrow \text{graphs.} \]

\[
\begin{pmatrix}
-3 & 3 & 0 \\
3 & -5 & 2 \\
0 & 2 & -2
\end{pmatrix}
\]

\[ G \left( \begin{pmatrix}
-3 & 3 & 0 \\
3 & -5 & 2 \\
0 & 2 & -2
\end{pmatrix} \right) \]
$G : \text{real symmetric matrices} \rightarrow \text{graphs}.$

$$
\begin{pmatrix}
-3 & 3 & 0 \\
3 & -5 & 2 \\
0 & 2 & -2
\end{pmatrix}
\xrightarrow{G}
$$

$S(G) = \{A \in M_{n \times n}(\mathbb{R}): A = A^t, G(A) = G\}.$
The **minimum rank** of a graph $G$ is

$$mr(G) = \min \{ \text{rank}(A): A \in S(G) \}.$$
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The maximum nullity of a graph $G$ is

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The minimum rank problem of a graph $G$ is to determine the number $\text{mr}(G)$ or $M(G)$.  

Applications of z. f. number to the minimum rank problem
The **zero forcing process** on a graph $G$ is the color-changing process using the following rules.
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- Each vertex of $G$ is either black or white initially.

A set $F \subseteq V(G)$ is called a zero forcing set if with the initial condition $F$, each vertex of $G$ could be forced into black.

The zero forcing number $Z(G)$ of a graph $G$ is the minimum size of a zero forcing set.

The path cover number $P(G)$ of a graph $G$ is the minimum number of vertex disjoint induced paths of $G$ that cover $V(G)$.
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The path cover number $P(G)$ of a graph $G$ is the minimum number of vertex disjoint induced paths of $G$ that cover $V(G)$. 
Example for Three Parameters

\[
\begin{pmatrix}
? & * & * & * \\
* & ? & 0 & 0 \\
* & 0 & ? & 0 \\
* & 0 & 0 & ?
\end{pmatrix}
\]

\[G\]

\[\text{rank} \geq 2.
\]

\[Z(G) = 2.\]

\[P(G) = 2.\]
Example for Three Parameters

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

\[G\]

- rank \(\geq 2\).
- 2 is achievable.
Example for Three Parameters

\[
\begin{pmatrix}
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1 & 0 & 0 & 0 \\
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\]

- \( \text{rank} \geq 2. \)
- 2 is achievable.
- \( \text{mr}(K_{1,3}) = 2 \) and \( \text{M}(K_{1,3}) = 4 - 2 = 2. \)
Example for Three Parameters

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
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\( G \)

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\(G\)

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- \(Z(G) = 2\).
- \(P(G) = 2\).
For all graph $G$, $M(G) \leq Z(G)$. [1]
Basic Properties

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- For outerplanar graph $G$, $M(G) \leq P(G) \leq Z(G).$[12]
Basic Properties

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For outerplanar graph $G$, $M(G) \leq P(G) \leq Z(G)$. [12]

$M(G)$ and $P(G)$ are not comparable in general.
A chronological list record the order of forces.

- **Chronological list**
  
  1 $\rightarrow$ 2 $\rightarrow$ 4 $\rightarrow$ 3
  
  5 $\rightarrow$ 6 $\rightarrow$ 8 $\rightarrow$ 10
  
  7 $\rightarrow$ 8 $\rightarrow$ 10 $\rightarrow$ 9
  
  6 $\rightarrow$ 4

- **Maximal chains**
  
  1 $\rightarrow$ 2
  
  5 $\rightarrow$ 6 $\rightarrow$ 4 $\rightarrow$ 3
  
  7 $\rightarrow$ 8 $\rightarrow$ 10 $\rightarrow$ 9
Terminologies for $Z(G)$

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- A chain of a chronological list is a sequence of consecutive forcing list.
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- A **chronological list** records the order of forces.
- A **chain** of a chronological list is a sequence of consecutive forcing list.
- The set of maximal chains forms a path cover.
- The inverse chronological list gives another zero forcing set called **reversal**.

![Diagram](image)
The vertex-sum of $G_1$ and $G_2$ at the vertex $v$ is the graph $G_1 \oplus_v G_2$ obtained by identifying the vertex $v$. 
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If $G = G_1 \oplus_v G_2$, then

$$M(G) = \max \{ M(G_1) + M(G_2) - 1, M(G_1 - v) + M(G_2 - v) - 1 \}.$$

\[4\]
A vertex \( v \) is **doubly terminal** if \( v \) is a one-vertex path in some optimal path cover.
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The **path spread** of $G$ on $v$ is

$$p_v(G) = P(G) - P(G - v).$$
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If $G = G_1 \oplus_v G_2$, then

$$p_v(G) = \begin{cases} 
-1, & \text{if } v \text{ is simply terminal of } G_1 \text{ and } G_2; \\
\min\{p_v(G_1), p_v(G_2)\}, & \text{otherwise.}[5]
\end{cases}$$
Reduction Formula for $Z(G)$

- A vertex $v$ is **doubly terminal** if $v$ is a one-vertex maximal chain in some optimal chronological list.
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If $G = G_1 \oplus_v G_2$, then

$$z_v(G) = \begin{cases} -1 & \text{if $v$ is simply terminal of $G_1$ and $G_2$;} \\ \min\{z_v(G_1), z_v(G_2)\} & \text{otherwise.} \end{cases}$$
Sketch of Proof

\[ -1 \leq z_v(G) \leq 1. \]
\[ v \text{ is doubly terminal} \iff z_v = 0. \]
\[ v \text{ is simply terminal} \implies z_v = 0. \]
Sketch of Proof

\[
-1 \leq z_v(G) \leq 1.
\]

\[
\begin{align*}
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If \( v \) is simply terminal for \( G_1 \) and \( G_2 \), then \( z_v(G) = -1 \),

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Sketch of Proof

If $G = G_1 \oplus_v G_2$, then

\[ Z(G) \leq Z(G_1) + Z(G_2 - v), \quad Z(G) \leq Z(G_1 - v) + Z(G_2), \]

\[ Z(G) \geq Z(G_1) + Z(G_2) - 1. \]
Sketch of Proof

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  $$Z(G) \geq Z(G_1) + Z(G_2) - 1.$$ 

- If $G = G_1 \oplus_v G_2$, then
  
  $$z_v(G) \leq \min\{z_v(G_1), z_v(G_2)\},$$
  
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  \]
  
  \[
  Z(G) \geq Z(G_1) + Z(G_2) - 1.
  \]

- If \( G = G_1 \oplus_v G_2 \), then
  
  \[
  z_v(G) \leq \min\{z_v(G_1), z_v(G_2)\},
  \]
  
  \[
  z_v(G) \geq z_v(G_1) + z_v(G_2) - 1.
  \]

- \( z_v(G) = -1, \ z_v(G_1) = z_v(G_2) = 0 \) is the only possibility. This implies \( v \) is simply terminal for \( G_1 \) and \( G_2 \).
Comparison of Reduction Formulae

Denote \( m_v(G) = M(G) - M(G - v) \), \( p_v(G) = P(G) - P(G - v) \), and \( z_v(G) = Z(G) - Z(G - v) \).
Denote $m_v(G) = M(G) - M(G - v)$, $p_v(G) = P(G) - P(G - v)$, and $z_v(G) = Z(G) - Z(G - v)$.

$-1 \leq m_v, p_v, r_v \leq 1$. 

Hard to apply on induction.

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Comparison of Reduction Formulae

- Denote $m_v(G) = M(G) - M(G - v)$, $p_v(G) = P(G) - P(G - v)$, and $z_v(G) = Z(G) - Z(G - v)$.
- $-1 \leq m_v, p_v, r_v \leq 1$.
- If $G = G_1 \oplus_v G_2$, they have similar behavior.

<table>
<thead>
<tr>
<th>$m_v(G_1 \setminus G_2)$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
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<td>1</td>
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<table>
<thead>
<tr>
<th>$p_v, z_v(G_1 \setminus G_2)$</th>
<th>-1</th>
<th>0</th>
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<tr>
<td>-1</td>
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<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>-1\0</td>
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Denote $m_v(G) = M(G) - M(G - v)$, 
$p_v(G) = P(G) - P(G - v)$, and $z_v(G) = Z(G) - Z(G - v)$.

$-1 \leq m_v, p_v, r_v \leq 1.$

If $G = G_1 \oplus_v G_2$, they have similar behavior.

$$
\begin{array}{cccc}
  m_v(G_1 \backslash G_2) & -1 & 0 & 1 \\
  -1 & -1 & -1 & -1 \\
  0 & -1 & -1 & 0 \\
  1 & -1 & 0 & 1 \\
\end{array}
$$

$$
\begin{array}{cccc}
  p_v, z_v(G_1 \backslash G_2) & -1 & 0 & 1 \\
  -1 & -1 & -1 & -1 \\
  0 & -1 & -1 & 0 \\
  1 & -1 & 0 & 1 \\
\end{array}
$$

Hard to apply on induction.
Recall that $P(G) \leq Z(G)$.
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A graph $G$ satisfies the PZ condition iff $P(G) = Z(G)$. 
The PZ condition

- Recall that $P(G) \leq Z(G)$.
- A graph $G$ satisfies the PZ condition iff $P(G) = Z(G)$.
- PZ condition is not hereditary.

\begin{center}
\begin{tikzpicture}
\node (1) at (1,2) [label=left:1] {};
\node (2) at (2,1) [label=left:2] {};
\node (3) at (3,1) [label=left:3] {};
\node (4) at (4,2) [label=left:4] {};
\node (5) at (3,0) [label=left:5] {};
\node (6) at (4,0) [label=left:6] {};
\draw (1) -- (2) -- (3) -- (4) -- (1);
\draw (5) -- (6) -- (4) -- (2) -- (5);
\end{tikzpicture}
\end{center}
The PZ condition

- Recall that \( P(G) \leq Z(G) \).
- A graph \( G \) satisfies the **PZ condition** iff \( P(G) = Z(G) \).
- PZ condition is not hereditary.
- PZ condition does not preserve under vertex-sum operation.

\[
G_1 \oplus_v G_2
\]

\[
G_1 ~ G_2 ~ G_1 \oplus_v G_2
\]
A graph $G$ satisfies the **strong PZ condition** iff each path cover is the set of maximal chain for some zero forcing process.
The Strong PZ condition

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Strong PZ condition preserves under vertex-sum operation.
A graph $G$ satisfies the **strong PZ condition** iff each path cover is the set of maximal chain for some zero forcing process.

Strong PZ condition $\Rightarrow$ PZ condition.

Strong PZ condition is hereditary.

Strong PZ condition preserves under vertex-sum operation.
A cactus is a graph whose blocks are all $K_2$ or $C_n$.
A **cactus** is a graph whose blocks are all $K_2$ or $C_n$.

A cactus $G$ satisfies the strong PZ condition. Hence we have $P(G) = Z(G)$. 
Let $G_k$ be the $k$ 5-sun sequence. Then

$$P(G_k) = Z(G_k) = 2k + 1 \text{ and } M(G_k) = k + 1.$$
Large $Z(G) - M(G)$

- Let $G_k$ be the $k$ 5-sun sequence. Then $P(G_k) = Z(G_k) = 2k + 1$ and $M(G_k) = k + 1$.
- Actually, for all $1 \leq p \leq q \leq 2p - 1$, there is a graph $G$ such that $M(G) = p$ and $Z(G) = q$. 

![Graphs](image.png)
Let $G_k$ be the $k$ 5-sun sequence. Then $P(G_k) = Z(G_k) = 2k + 1$ and $M(G_k) = k + 1$.

Actually, for all $1 \leq p \leq q \leq 2p - 1$, there is a graph $G$ such that $M(G) = p$ and $Z(G) = q$.

Q: Will the inequality $Z(G) \leq 2M(G) - 1$ holds for all $G$?
A sign set is \{0, *, u\}. A real number \( r \) matches 0 if \( r = 0 \), * if \( r \neq 0 \), while \( u \) if \( r \) matches 0 or *. 
A sign set is \( \{0, *, u\} \). A real number \( r \) matches 0 if \( r = 0 \), * if \( r \neq 0 \), while \( u \) if \( r \) matches 0 or *. 

A pattern matrix \( Q \) is a matrix over \( S \).
A sign set is \( \{0, *, u\} \). A real number \( r \) matches 0 if \( r = 0 \), \( * \) if \( r \neq 0 \), while \( u \) if \( r \) matches 0 or \( * \).

A pattern matrix \( Q \) is a matrix over \( S \).

The minimum rank of a pattern \( Q \) is

\[
\text{mr}(Q) = \min\{\text{rank}A : A \cong Q\}.
\]
Example for Minimum Rank of A Pattern

The pattern

\[ Q = \begin{pmatrix} * & 0 & 0 \\ u & * & u \end{pmatrix} \]

must have rank at least 2.
The pattern

\[ Q = \begin{pmatrix} * & 0 & 0 \\ u & * & u \end{pmatrix} \]

must have rank at least 2.

The rank 2 is achievable. Hence \( \text{mr}(Q) = 2 \).
Define addition “+” and scalar multiplication “×” on $S$.

\[ +: S \times S \rightarrow S \]

\[ \begin{array}{c|ccc}
+ & 0 & * & u \\
\hline
0 & 0 & * & u \\
* & * & u & u \\
u & u & u & u \\
\end{array} \]

\[ \times: \{0, *\} \times S \rightarrow S \]

\[ \begin{array}{c|ccc}
\times & 0 & * & u \\
\hline
0 & 0 & 0 & 0 \\
* & 0 & * & u \\
\end{array} \]
A sign vector is a tuple with entries on $S$. 

A set of sign vectors $\{v_1, v_2, \ldots, v_n\}$ is independent iff $c_1v_1 + c_2v_2 + \cdots + c_nv_n \sim 0$ implies $c_1 = c_2 = \cdots = c_n = 0$. 

The rank of a pattern is the maximum number of independent row sign vectors.
Independence

- A **sign vector** is a tuple with entries on $S$.
- We say a sign vector $v \sim 0$ iff $v$ contains no $\ast$. 
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$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \sim 0$$

implies $c_1 = c_2 = \cdots = c_n = 0$. 

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Independence

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- We say a sign vector $v \sim 0$ iff $v$ contains no $\ast$.
- A set of sign vectors $\{v_1, v_2, \ldots, v_n\}$ is **independent** iff

  \[ c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \sim 0 \]

  implies $c_1 = c_2 = \cdots = c_n = 0$.
- The **rank** of a pattern is the maximum number of independent row sign vectors.
Independence in different senses

Lemma

Suppose $V = \{v_1, v_2, \ldots, v_n\}$ is a set of sign vectors, and $W = \{w_1, w_2, \ldots, w_n\}$ is a set of sign vectors such that $w_i$ is obtained from $v_i$ by replacing entries $u$ by $0$ or $\ast$. If $V$ is linearly independent, then so is $W$.

Suppose $R = \{r_1, r_2, \ldots, r_n\}$ is a set of real vectors such that each entry in each vector matches the corresponding entry in elements of $W$. If $W$ is linearly independent, then $R$ is linearly independent as real vectors.
Independence in different senses

**Lemma**

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Suppose $R = \{r_1, r_2, \ldots, r_n\}$ is a set of real vectors such that each entry in each vector matches the corresponding entry in elements of $W$. If $W$ is linearly independent, then $R$ is linearly independent as real vectors.

**Theorem**

If $Q$ is a pattern matrix and $U$ is the set of all pattern matrices obtained from $Q$ by replacing $u$ by 0 or $\ast$, then

$$\text{rank}(Q) \leq \min_{Q' \in U} \{\text{rank}(Q')\} \leq \text{mr}(Q).$$
Let $G$ be a graph and $B$ is a subset of $E(G)$ called the set of **banned edge** or **banned set**.
Let $G$ be a graph and $B$ is a subset of $E(G)$ called the set of banned edge or banned set.
The zero forcing process on $G$ banned by $B$ is the coloring process by following rules.

- Each vertex of $G$ is either black or white initially.
- If $x$ is a black vertex and $y$ is the only white neighbor of $x$ and $xy \notin B$, then change the color of $y$ to black.

Zero forcing set banned by $B$ $F$ can force $V(G)$ banned by $B$.

Zero forcing number banned by $B$ $Z(G, B)$: minimum size of $F$.

Zero forcing number banned by $B$ with support $W$ $Z(W, B)$: minimum size of $F \supseteq W$.

When $W$ and $B$ is empty, $Z(W, B) = Z(G)$.
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Applications of z. f. number to the minimum rank problem
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Applications of z. f. number to the minimum rank problem
Q is a given $m \times n$ pattern. $G = (X \cup Y, E)$ is the related bipartite defined by

$$X = \{a_1, a_2, \ldots, a_m\}, \ Y = \{b_1, b_2, \ldots, b_n\}, \ E = \{a_i b_j : Q_{ij} \neq 0\}.$$
- $Q$ is a given $m \times n$ pattern. $G = (X \cup Y, E)$ is the related bipartite defined by

  $X = \{a_1, a_2, \ldots, a_m\}$, $Y = \{b_1, b_2, \ldots, b_n\}$, $E = \{a_i b_j : Q_{ij} \neq 0\}$.

- $B = \{a_i b_j : Q_{ij} = u\}$.

\[
\begin{pmatrix}
* & 0 & 0 \\
u & * & u
\end{pmatrix}
\]
Main Theorem

**Theorem**

For a given $m \times n$ pattern matrix $Q$, if $G = (X \cup Y, E)$ is the graph and $B$ is the set of banned edges defined above, then

$$\text{rank}(Q) + Z_Y(G, B) = m + n.$$
Theorem

For a given $m \times n$ pattern matrix $Q$, if $G = (X \cup Y, E)$ is the graph and $B$ is the set of banned edges defined above, then

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- Each initial white vertex represent a sign vector.
Theorem

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- Each initial white vertex represent a sign vector.
- The set of initial white vertices is independent iff it will be forced.
Main Theorem

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Main Theorem

**Theorem**

For a given $m \times n$ pattern matrix $Q$, if $G = (X \cup Y, E)$ is the graph and $B$ is the set of banned edges defined above, then

$$\text{rank}(Q) + Z_Y(G, B) = m + n.$$ 

- Each initial white vertex represents a sign vector.
- The set of initial white vertices is independent iff it will be forced.

\[
\begin{pmatrix}
  u & 0 \\
  * & * \\
  0 & u
\end{pmatrix}
\]

\[
\begin{pmatrix}
  u \\
  * \\
  0
\end{pmatrix} + 
\begin{pmatrix}
  * \\
  u \\
  *
\end{pmatrix} + 
\begin{pmatrix}
  0 \\
  * \\
  u
\end{pmatrix} = 
\begin{pmatrix}
  u \\
  u \\
  u
\end{pmatrix} \sim 0
\]
Recall that $\text{rank}(Q) \leq \min_{Q' \in U}\{\text{rank}(Q')\} \leq \text{mr}(Q)$. The middle term is called the **exhaustive rank** of $Q$. 

For a given graph $G$, there is a corresponding pattern $Q$ whose diagonal entries are all $u$. Let $I \subseteq [n]$ and $Q_I$ be the pattern replace those $u$ in $ii$-entry by $\ast$ if $i \in I$ and $0$ if $i \notin I$. Then $U = \{Q_I : I \subseteq [n]\}$. Define $\tilde{G}_I$ to be the bipartite given by $Q_I$. The inequality become $M(G) \leq \max_{I \subseteq [n]} Z_Y(\tilde{G}_I) - n \leq Z_Y(\tilde{G}_{[n]}, B) - n$.

The second term is called the **exhaustive zero forcing number** of $G$. Denote it by $\tilde{Z}(G)$. The third term could be proven to equal $Z(G)$. Hence $M(G) \leq \tilde{Z}(G) \leq Z(G)$. 

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Applications of z. f. number to the minimum rank problem
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For a given graph $G$, there is a corresponding pattern $Q$ whose diagonal entries are all $u$. 

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Let $I \subseteq [n]$ and $Q_I$ be the pattern replace those $u$ in $ii$-entry by $\ast$ if $i \in I$ and 0 if $i \notin I$. Then $U = \{ Q_I : I \subseteq [n] \}$. Define $\tilde{G}_I$ to be the bipartite given by $Q_I$. The inequality becomes

$$M(G) \leq \max_{I \subseteq [n]} Z_{\tilde{Y}}(\tilde{G}_I) - n \leq Z_{\tilde{Y}}(\tilde{G}_{[n]}, B) - n.$$
The Exhaustive Zero Forcing Number

- Recall that $\text{rank}(Q) \leq \min_{Q' \in U} \{\text{rank}(Q')\} \leq \text{mr}(Q)$. The middle term is called the **exhaustive rank** of $Q$.
- For a given graph $G$, there is a corresponding pattern $Q$ whose diagonal entries are all $u$.
- Let $I \subseteq [n]$ and $Q_I$ be the pattern replace those $u$ in $ii$-entry by $\ast$ if $i \in I$ and 0 if $i \notin I$. Then $U = \{Q_I : I \subseteq [n]\}$. Define $\tilde{G}_I$ to be the bipartite given by $Q_I$.
- The inequality become

$$M(G) \leq \max_{I \subseteq [n]} Z_{\gamma}(\tilde{G}_I) - n \leq Z_{\gamma}(\tilde{G}_{[n]}, B) - n.$$
Recall that \( \text{rank}(Q) \leq \min_{Q' \in U} \{\text{rank}(Q')\} \leq \text{mr}(Q) \). The middle term is called the exhaustive rank of \( Q \).

For a given graph \( G \), there is a corresponding pattern \( Q \) whose diagonal entries are all \( u \).

Let \( I \subseteq [n] \) and \( Q_I \) be the pattern replace those \( u \) in \( ii \)-entry by \( \ast \) if \( i \in I \) and 0 if \( i \notin I \). Then \( U = \{ Q_I : I \subseteq [n] \} \). Define \( \tilde{G}_I \) to be the bipartite given by \( Q_I \).

The inequality become

\[
M(G) \leq \max_{I \subseteq [n]} Z_Y(\tilde{G}_I) - n \leq Z_Y(\tilde{G}_{[n]}, B) - n.
\]

The second term is called the exhaustive zero forcing number of \( G \). Denote it by \( \tilde{Z}(G) \). The third term could be proven to equal \( Z(G) \).
Recall that $\text{rank}(Q) \leq \min_{Q' \in U} \{\text{rank}(Q')\} \leq \text{mr}(Q)$. The middle term is called the \textit{exhaustive rank} of $Q$.

For a given graph $G$, there is a corresponding pattern $Q$ whose diagonal entries are all $u$.

Let $I \subseteq [n]$ and $Q_I$ be the pattern replace those $u$ in $ii$-entry by $*$ if $i \in I$ and $0$ if $i \notin I$. Then $U = \{Q_I : I \subseteq [n]\}$. Define $\tilde{G}_I$ to be the bipartite given by $Q_I$.

The inequality become

$$M(G) \leq \max_{I \subseteq [n]} Z_Y(\tilde{G}_I) - n \leq Z_Y(\tilde{G}_{[n]}, B) - n.$$  

The second term is called the \textit{exhaustive zero forcing number} of $G$. Denote it by $\tilde{Z}(G)$. The third term could be proven to equal $Z(G)$.

Hence $M(G) \leq \tilde{Z}(G) \leq Z(G)$. 
For $G = P_3$, the pattern is

$$Q = \begin{pmatrix} u & * & 0 \\ * & u & * \\ 0 & * & u \end{pmatrix}.$$
• For $G = P_3$, the pattern is

$$Q = \begin{pmatrix} u & * & 0 \\ * & u & * \\ 0 & * & u \end{pmatrix}.$$ 

• For $I = \{1, 3\} \subseteq [3]$, the pattern is

$$Q = \begin{pmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{pmatrix}.$$
• For $G = P_3$, the pattern is

$$Q = \begin{pmatrix} u & * & 0 \\ * & u & * \\ 0 & * & u \end{pmatrix}.$$ 

• For $I = \{1, 3\} \subseteq [3]$, the pattern is

$$Q = \begin{pmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{pmatrix}.$$ 

• $1 = M(P_3) \leq \tilde{Z}(P_3) \leq Z(P_3) = 1$. Hence $\tilde{Z}(G) = 1$. 

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Applications of z. f. number to the minimum rank problem
Bipartites related to $P_3$

Applications of z. f. number to the minimum rank problem
Theorem

If $G$ is the bipartite given by a pattern $Q$, then

$$Z_Y(G, B) = Z_X(G, B).$$

- **Row rank**: maximum number of rows; **Column rank**: maximum number of columns.
Theorem

If $G$ is the bipartite given by a pattern $Q$, then

$$Z_Y(G, B) = Z_X(G, B).$$

- **Row rank**: maximum number of rows; **Column rank**: maximum number of columns.
- **Row rank** = **Column rank**!
The *n*-sun is a graph obtained by adding *n* leaves to each vertices of $C_n$. 
The *n*-sun

- The *n*-sun is a graph obtained by adding *n* leaves to each vertex of $C_n$.
- In [4], it was shown $M(H_3) = Z(H_3) = 2$ and $M(H_n) = \lfloor \frac{n}{2} \rfloor$, $Z(H_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 4$. 
The $n$-sun is a graph obtained by adding $n$ leaves to each vertices of $C_n$.

In [4], it was shown $M(H_3) = Z(H_3) = 2$ and $M(H_n) = \left\lfloor \frac{n}{2} \right\rfloor$, $Z(H_n) = \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 4$.

But $M(G) = \tilde{Z}(H_n)$ for all $n \geq 3$!
The $n$-sun is a graph obtained by adding $n$ leaves to each vertices of $C_n$.

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But $M(G) = \tilde{Z}(H_n)$ for all $n \geq 3$!

The computation could either by discussion on the patterns of those leaves or by the sieving process given below.
The $n$-sun

- The $n$-sun is a graph obtained by adding $n$ leaves to each vertices of $C_n$.
- In [4], it was shown $M(H_3) = Z(H_3) = 2$ and $M(H_n) = \left\lceil \frac{n}{2} \right\rceil$, $Z(H_n) = \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 4$.
- But $M(G) = \tilde{Z}(H_n)$ for all $n \geq 3$!
- The computation could either by discussion on the patterns of those leaves or by the sieving process given below.
- The parameter $\tilde{Z}(G)$ is still not sharp for some cactus.
Example for Sieving Process

If $Z(\overline{H_{5l}}) - 10 = 3$ for some $l$, then $1 \in l$ and $2 \notin l$, a contradiction.

$$\tilde{Z}(G) = 12 - 10 = 2.$$
If \( Z(\overline{H_5}) - 10 = 3 \) for some \( I \), then \( 1 \in I \) and \( 2 \notin I \), a contradiction.

\[ \tilde{Z}(G) = 12 - 10 = 2. \]
Example for Sieving Process

- If $Z(\widetilde{H_{5,l}}) - 10 = 3$ for some $l$, then $1 \in l$ and $2 \notin l$, a contradiction.

$$\tilde{Z}(G) = 12 - 10 = 2.$$
Example for Sieving Process

If $Z(\tilde{\mathcal{H}_{5l}}) - 10 = 3$ for some $l$, then $1 \in l$ and $2 \notin l$, a contradiction.

$$
\tilde{Z}(G) = 12 - 10 = 2.
$$
Example for Sieving Process

- If $Z(\widehat{H}_{5I}) - 10 = 3$ for some $I$, then $1 \in I$ and $2 \notin I$, a contradiction.

$$\tilde{Z}(G) = 12 - 10 = 2.$$
Example for Sieving Process

If $Z(\widetilde{H}_5 I) - 10 = 3$ for some $I$, then $1 \in I$ and $2 \notin I$, a contradiction.

$\widetilde{Z}(G) = 12 - 10 = 2$. 

[Diagram of graphs showing vertices labeled 1 to 5 with arrows indicating direction]
Example for Sieving Process

- If $Z(\widehat{H_{5I}}) - 10 = 3$ for some $I$, then $1 \in I$ and $2 \notin I$, a contradiction.

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Example for Sieving Process

If \( Z(\tilde{H}_{5I}) - 10 = 3 \) for some \( I \), then \( 1 \in I \) and \( 2 \notin I \), a contradiction.

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Example for Sieving Process

- If $\tilde{Z}(\tilde{\mathcal{H}}_{5,l}) - 10 = 3$ for some $l$, then $1 \in l$ and $2 \notin l$, a contradiction.

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Example for Sieving Process

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- $\tilde{Z}(G) = 12 - 10 = 2$. 
Example for Sieving Process

If $Z(\widetilde{H}_{5I}) - 10 = 3$ for some $I$, then $1 \in I$ and $2 \notin I$, a contradiction.

$$\widetilde{Z}(G) = 12 - 10 = 2.$$
Example for Sieving Process

If $Z(\overline{H}_{5I}) - 10 = 3$ for some $I$, then $1 \in I$ and $2 \notin I$, a contradiction.

$\tilde{Z}(G) = 12 - 10 = 2.$
If $\tilde{Z}(H_{5,l}) - 10 = 3$ for some $l$, then $1 \in l$ and $2 \notin l$, a contradiction.

$\tilde{Z}(G) = 12 - 10 = 2$. 
Edge vs Nonedge

- Edge: **Increase** number of neighbor; **Increase** possible route for passing.

- Nonedge: **Decrease** number of neighbor; **Decrease** possible route for passing.

The BAD guy Banned

Edge: **Increase** number of neighbor; **Decrease** possible route for passing.

---

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Applications of z. f. number to the minimum rank problem
Edge vs Nonedge

- **Edge**: *Increase* number of neighbor; *Increase* possible route for passing.
- **Nonedge**: *Decrease* number of neighbor; *Decrease* possible route for passing.

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Applications of z. f. number to the minimum rank problem
Edge vs Nonedge

- **Edge**: *Increase* number of neighbor; *Increase* possible route for passing.
- **Nonedge**: *Decrease* number of neighbor; *Decrease* possible route for passing.
- The BAD guy Banned Edge: *Increase* number of neighbor; *Decrease* possible route for passing.
Rewrite

\[ \tilde{Z}(G) = \max_{I \subseteq [n]} Z_Y(\tilde{G}_I) - n = \max \{ k : k = Z_Y(\tilde{G}_I) - n \text{ for some } I \}. \]
Rewrite

\[ \tilde{Z}(G) = \max_{I \subseteq [n]} Z_Y(\tilde{G}_I) - n = \max\{ k : k = Z_Y(\tilde{G}_I) - n \text{ for some } I \} . \]

Let \( \mathcal{I}_k(G) = \{ I \subseteq [n] : Z_Y(\tilde{G}_I) - n \geq k \} \).
Rewrite

\[ \tilde{Z}(G) = \max_{l \subseteq [n]} Z_Y(\tilde{G}_l) - n = \max \{ k : k = Z_Y(\tilde{G}_l) - n \text{ for some } l \} . \]

Let \( \mathcal{I}_k(G) = \{ l \subseteq [n] : Z_Y(\tilde{G}_l) - n \geq k \} \).

\[ \tilde{Z}(G) = \max \{ k : \mathcal{I}_k \neq \emptyset \} . \]
Rewrite

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\[ \tilde{Z}(G) = \max \{ k : \mathcal{I}_k \neq \emptyset \} . \]

Each \( F \supseteq Y \) with size \( n + k - 1 \) is a sieve for \( \mathcal{I}_k(G) \) to delete impossible index sets.
Nonzero-vertex and Zero-vertex

- If \( i \in l \) for all \( l \in \mathcal{I}_k(G) \), then \( i \) is called a nonzero-vertex.
If \( i \in l \) for all \( l \in \mathcal{I}_k(G) \), then \( i \) is called a **nonzero-vertex**.

If \( i \notin l \) for all \( l \in \mathcal{I}_k(G) \), then \( i \) is called a **zero-vertex**.
If \( i \in I \) for all \( I \in \mathcal{I}_k(G) \), then \( i \) is called a \textbf{nonzero-vertex}.

If \( i \notin I \) for all \( I \in \mathcal{I}_k(G) \), then \( i \) is called a \textbf{zero-vertex}.

Each leaf in \( H_5 \) is a zero-vertex and nonzero-vertex in \( \mathcal{I}_3(H_5) \) simultaneously. Hence \( \mathcal{I}_3(H_5) = \emptyset \).
Nonzero-vertex and Zero-vertex

- If $i \in I$ for all $I \in \mathcal{I}_k(G)$, then $i$ is called a nonzero-vertex.
- If $i \notin I$ for all $I \in \mathcal{I}_k(G)$, then $i$ is called a zero-vertex.
- Each leaf in $H_5$ is a zero-vertex and nonzero-vertex in $\mathcal{I}_3(H_5)$ simultaneously. Hence $\mathcal{I}_3(H_5) = \emptyset$.
- For $G = K_n$, each vertex is a nonzero-vertex in $\mathcal{I}_{n-1}(G)$ for $n \geq 2$ while a zero-vertex in $\mathcal{I}_1(G)$.

Applications of z. f. number to the minimum rank problem
Nonzero-vertex and Zero-vertex

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- For \( G = K_n \), each vertex is a nonzero-vertex in \( \mathcal{I}_{n-1}(G) \) for \( n \geq 2 \) while a zero-vertex in \( \mathcal{I}_1(G) \).
- For \( G = K_{1,t} \), \( t \geq 2 \), each leaf is a zero-vertex in \( \mathcal{I}_{t-1}(G) \).
Nonzero-vertex and Zero-vertex

If \( i \in I \) for all \( I \in \mathcal{I}_k(G) \), then \( i \) is called a nonzero-vertex.

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Each leaf in \( H_5 \) is a zero-vertex and nonzero-vertex in \( \mathcal{I}_3(H_5) \) simultaneously. Hence \( \mathcal{I}_3(H_5) = \emptyset \).

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For \( G = K_{1,t}, t \geq 2 \), each leaf is a zero-vertex in \( \mathcal{I}_{t-1}(G) \).

For multi-partite \( G \) with more than one part and more than two vertices in each parts, each vertex is a zero-vertex in \( \mathcal{I}_{n-2}(G), n = |V(G)| \).
Nonzero-vertex and Zero-vertex

- If $i \in I$ for all $I \in \mathcal{I}_k(G)$, then $i$ is called a **nonzero-vertex**.
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Each leaf in $H_5$ is a zero-vertex and nonzero-vertex in $\mathcal{I}_3(H_5)$ simultaneously. Hence $\mathcal{I}_3(H_5) = \emptyset$.

For $G = K_n$, each vertex is a nonzero-vertex in $\mathcal{I}_{n-1}(G)$ for $n \geq 2$ while a zero-vertex in $\mathcal{I}_1(G)$.

For $G = K_{1,t}$, $t \geq 2$, each leaf is a zero-vertex in $\mathcal{I}_{t-1}(G)$.

For multi-partite $G$ with more than one part and more than two vertices in each parts, each vertex is a zero-vertex in $\mathcal{I}_{n-2}(G)$, $n = |V(G)|$.

We know $Z(G_k) = 2k + 1$ and $M(G_k) = k + 1$. By sieving process, $\tilde{Z}(G_k) = k + 1!$ Here $G_k$ is the $k$ 5-sun sequence.
Example for Stronger Upper Bound 1

- $M(G) \leq Z(G) = 7$. Each vertex is a zero-vertex in $\mathcal{I}_7$. 

![Graph Image]
Example for Stronger Upper Bound 1

- $M(G) \leq Z(G) = 7$. Each vertex is a zero-vertex in $\mathcal{I}_7$.
- If $A \in S(G)$ has nullity 7, we may assume

$$A = \begin{pmatrix} O & J & J \\ J & O & B^t \\ J & B & O \end{pmatrix}.$$

The matrix $A$ has the same nullity 7. It is impossible when $char \neq 2$.

- $M(G) \leq 6$. And actually $M(G) = 6$.  

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Example for Stronger Upper Bound 1

- $M(G) \leq Z(G) = 7$. Each vertex is a zero-vertex in $I_7$.
- If $A \in S(G)$ has nullity 7, we may assume
  
  $$A = \begin{pmatrix} O & J & J \\ J & O & B^t \\ J & B & O \end{pmatrix}.$$ 

- The matrix
  
  $$\begin{pmatrix} O & J & O \\ J & O & B^t \\ O & B & -B - B^t \end{pmatrix}$$

  has the same nullity 7.
Example for Stronger Upper Bound 1

- \( M(G) \leq Z(G) = 7 \). Each vertex is a zero-vertex in \( I_7 \).
- If \( A \in S(G) \) has nullity 7, we may assume
  \[
  A = \begin{pmatrix}
  O & J & J \\
  J & O & B^t \\
  J & B & O
  \end{pmatrix}.
  \]

- The matrix
  \[
  \begin{pmatrix}
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  J & O & B^t \\
  O & B & -B - B^t
  \end{pmatrix}
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  has the same nullity 7.
- \(-B - B^t = O\). It is impossible when \text{char} \neq 2.
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- $-B - B^t = O$. It is impossible when char $\neq 2$.
- $M(G) \leq 6$. And actually $M(G) = 6$. 

![Graph diagram]
Theorem

For a graph $G$, suppose $i$ is a nonzero-vertex in $\mathcal{I}_k(G)$. And $\eta_i(G)$ denote the set of those graphs obtained from $G$ by the following rules:

- The vertex $i$ should be deleted;
- For any neighbors $x$ and $y$ of $i$, the pair $xy$ should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$.

If the nullity $k$ is achievable by some matrix in $S(G)$, then

$$k \leq \max\{M(H) : H \in \eta_i(G)\}.$$
If $k$ is achievable by $A \in \mathcal{S}(G)$, assume

$$A = \begin{pmatrix} 1 & a^t & 0 \\ a & \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}.$$

The nullity of $A$ should be less than the maximum nullity of each possible matrix $P$. 

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Applications of z. f. number to the minimum rank problem
If $k$ is achievable by $A \in \mathcal{S}(G)$, assume

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Sketch of Proof

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- The nullity of $A$ should be less than the maximum nullity of each possible matrix $P$. 

For a graph $G$, suppose $i$ is a zero-vertex in $\mathcal{I}_k(G)$ and $j$ is a neighbor of $i$. Let

$$N_1 = \{v: iv \in E(G), v \neq j\}, \quad N_2 = \{v: jv \in E(G), iv \notin E(G), v \neq i\}.$$

And $\eta_{i \rightarrow j}(G)$ denote the set of those graphs obtained from $G$ by the following rules:

- The vertex $i$ and $j$ should be deleted;
- For $x \in N_1$ and $y \in N_2$, the pair $xy$ should be an edge if $xy \notin E(G)$ and could be an edge or a non-edge if $xy \in E(G)$;
- For $x$ and $y$ in $N_1$, the pair $xy$ could be an edge or a non-edge.

If the nullity $k$ is achievable by some matrix in $S(G)$, then

$$k \leq \max\{M(H): H \in \eta_{i \rightarrow j}(G)\}.$$
If $k$ is achievable by $A \in \mathcal{S}(G)$, assume

$$A = \begin{pmatrix} \alpha & a^t & O \\ a & \hat{A}_{11} & \hat{A}_{12} \\ O & \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}.$$ 

Here $\alpha$ has the form \( \begin{pmatrix} 0 & * \\ * & u \end{pmatrix} \) and $\alpha^{-1}$ has the form \( \begin{pmatrix} u & * \\ * & 0 \end{pmatrix} \).
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Example for Stronger Upper Bound 2

\[ \tilde{Z}(G) = Z(G) = P(G) = 3. \]
Example for Stronger Upper Bound 2

- $\tilde{Z}(G) = Z(G) = P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_3$. 

![Graph with vertices labeled 1 to 15 and edges connecting them.](image-url)
Example for Stronger Upper Bound 2

- $\tilde{Z}(G) = Z(G) = P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $I_3$.
- $G - 1$ is the only graph in $\eta_1(G)$.

![Graph Image]
\[ \tilde{Z}(G) = Z(G) = P(G) = 3. \]

- The vertex 1 is a nonzero-vertex in \( I_3 \).
- \( G - 1 \) is the only graph in \( \eta_1(G) \).
- If 3 is achievable, then \( 3 \leq M(G - 1) \leq 2 \), a contradiction.

Hence \( M(G) \leq 2 \).
\[ Z(G) = 4 \text{ and } P(G) = 3. \]
Example for Stronger Upper Bound 3

- $Z(G) = 4$ and $P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_4$. 
Example for Stronger Upper Bound 3

- $Z(G) = 4$ and $P(G) = 3$.
- The vertex 1 is a nonzero-vertex in $\mathcal{I}_4$.
- Let $e = 23$. Then $G - 1$ and $G - 1 - e$ are the only two graphs in $\eta_1(G)$.
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• The vertex 1 is a nonzero-vertex in $I_4$.
• Let $e = 23$. Then $G - 1$ and $G - 1 - e$ are the only two graphs in $\eta_1(G)$.
• If 4 is achievable, then $4 \leq \max\{M(G - 1), M(G - 1 - e)\} \leq 3$, a contradiction. Hence $M(G) \leq 3$. 
\[ Z(G) = P(G) = 5. \]
Example for Stronger Upper Bound 4

- $Z(G) = P(G) = 5$.
- The vertex 1 is a zero-vertex.
Example for Stronger Upper Bound 4

- \( Z(G) = P(G) = 5. \)
- The vertex 1 is a zero-vertex.
- \( \eta_{1\rightarrow 2}(G) \) contains only one graph \( H \).
Example for Stronger Upper Bound 4

- $Z(G) = P(G) = 5$.
- The vertex 1 is a zero-vertex.
- $\eta_{1\rightarrow 2}(G)$ contains only one graph $H$.
- If 5 is achievable, then $5 \leq M(H) \leq 4$, a contradiction. Hence $M(G) \leq 4$. 

![Diagram of graphs $G$ and $\eta_{1\rightarrow 2}(G)$]
Example for Stronger Upper Bound 5

- $Z(G) = P(G) = 6$. 

The vertex 5 is a nonzero-vertex. List $\eta_1(G)$. $P(G_i) \leq 5$ for $i = 1, 2, 3, 4$. And they are outerplanar. $M(G_5) = 5$ by reduction formula. $M(G_4) \leq 5$ by doing nonzero elimination lemma again on 1. If 6 is achievable, then $6 \leq 5$, a contradiction. Hence $M(G) \leq 5$. 

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- If 6 is achievable, then $6 \leq 5$, a contradiction. Hence $M(G) \leq 5$. 

[Diagrams of graphs]
Corollary

If $i$ is a vertex of a graph $G$ and $j$ is a neighbor of $i$, then

$$M(G) \leq \max\{M(H) : H \in \eta_i(G) \cup \eta_{i \rightarrow j}(G)\}.$$
A looped graph is a graph that allows loops. A vertex $x$ is a neighbor of itself if and only if there is a loop on it.

The enhanced zero forcing number $\tilde{Z}(G)$ is the maximum of $Z(\tilde{G})$ over all looped graphs $\tilde{G}$ obtained from $G$ by adding loops on vertices of $G$.

$M(G) \leq \tilde{Z}(G) \leq Z(G)$. [9]

Theorem

$\tilde{Z}(G) = \hat{Z}(G)$ for all graph $G$. 

Applications of z. f. number to the minimum rank problem
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The zero forcing process on a looped graph $\widehat{G}$ is the coloring process with the following rules:

- Each vertex of $\widehat{G}$ is either black or white initially.
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**Theorem**

$\tilde{Z}(G) = \hat{Z}(G)$ for all graph $G$. 
A **t-triangle** of $Q$ is a $t \times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero.

**Theorem**

$$\text{rank}(Q) = \text{tri}(Q) \text{ for all pattern } Q.$$
A \textit{t-triangle} of $Q$ is a $t \times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero.

The \textit{triangular number} of pattern $Q$, denote by $\text{tri}(Q)$, is the maximum size of triangle in $Q$.

\textbf{Theorem}

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$\text{mr}(Q) \geq \text{tri}(Q)$.

\textbf{Theorem}

$\text{rank}(Q) = \text{tri}(Q)$ \textit{for all pattern} $Q$. 
The edge spread of zero forcing number on an edge $e$ is
\[ z_e(G) = Z(G) - Z(G - e). \]
The edge spread of zero forcing number on an edge $e$ is $z_e(G) = Z(G) - Z(G - e)$.

Theorem 2.21 in [7] says that if $z_e(G) = -1$, then for every optimal zero forcing chain set of $G$, $e$ is an edge in a chain.
The edge spread of zero forcing number on an edge $e$ is 
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Theorem 2.21 in [7] says that if \( z_e(G) = -1 \), then for every optimal zero forcing chain set of $G$, $e$ is an edge in a chain.

Question 2.22 in [7] ask whether the converse of Theorem 2.21 is true.
The Counterexample

- $T$ is the turtle graph. $G = (X \cup Y, E)$ is construct from $T$ by

  \[ X = \{a_1, a_2, \ldots, a_{14}\}, \quad Y = \{b_1, b_2, \ldots, b_{14}\}, \]

  and

  \[ E(G) = E_1 \cup E_2, \]

  where

  \[ E_1 = \{a_i a_j: i \neq j\} \cup \{b_i b_j: i \neq j\}, \quad E_2 = \{a_i b_j: ij \in E(T) \text{ or } i = j\}. \]
Each optimal zero forcing set of $G$ is of the forms:

- $F_0 = Y \cup \{u, v\}$, where $u$ could be $a_3$ or $a_4$ and $v$ could be $a_6$ or $a_7$.
- $\{a_3, a_4, p\} \cup (Y - y)$ or $\{a_6, a_7, q\} \cup (Y - y)$, where $p$ could be $a_6$ or $a_7$, $q$ could be $a_3$ or $a_4$, and $y$ is an arbitrarily vertex in $Y$.
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- Each optimal zero forcing set of $G$ is of the forms:
  - $F_0$ or its automorphism types. $F_0 = Y \cup \{u, v\}$, where $u$ could be $a_3$ or $a_4$ and $v$ could be $a_6$ or $a_7$. 

Diagram:

- The edge $e = a_1 b_1$ is used in each optimal zero forcing set.
- $Z(G) = Z(G - e) = 16$ and so $z_e(G) = 0 \neq -1$. 

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![Graph with vertices and edges labeled 1 to 14]
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The edge $e = a_1b_1$ is used in each optimal zero forcing set. But $Z(G) = Z(G - e) = 16$ and so $z_e(G) = 0 \neq -1$. 

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Further Goals for The Minimum Rank Problem

- Reduction formula on $k$-separate.
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- Reduction formula on $k$-separate.
- Reduction Formula for $\tilde{Z}(G)$.

Symmetry condition was seldom used. There must be some parameter between $\tilde{Z}(G)$ and $M(G)$ and it is sharp for cactus graphs. Sym and Not Sym is different!

$$mr(Q) = 3$$ if Sym while $$mr(Q) = 2$$ if Not Sym.

The proof in [13] of $M(C_n) = 2$ could be generalized.

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C. J. Edholm, L. Hogben, M. Huynh, J. LaGrange, and D. D. Row, Vertex and edge spread of zero forcing number, maximum nullity, and maximum rank of a graph, *Hogben’s Homepage*.


