國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第一次期中考 March 25, 2024 **Midterm 1**

姓名 Name : Solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Let

$$A = \begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{y} & - \\ - & \mathbf{z} & - \end{bmatrix}, \quad B = \begin{bmatrix} - & \mathbf{y} & - \\ - & \mathbf{x} & - \\ - & \mathbf{z} & - \end{bmatrix}, \quad C = \begin{bmatrix} - & 2\mathbf{x} & - \\ - & 3\mathbf{y} & - \\ - & 4\mathbf{z} & - \end{bmatrix},$$

$$D = \begin{bmatrix} - & \mathbf{x} & - \\ - & 3\mathbf{x}+\mathbf{y} & - \\ - & 4\mathbf{x}+\mathbf{z} & - \end{bmatrix}, \quad E = \begin{bmatrix} - & \mathbf{x}+2\mathbf{y} & - \\ - & \mathbf{y}+2\mathbf{z} & - \\ - & \mathbf{z}+2\mathbf{x} & - \end{bmatrix}, \quad F = \begin{bmatrix} - & \mathbf{x}+\mathbf{z} & - \\ - & \mathbf{y} & - \\ - & \mathbf{x}+\mathbf{z} & - \end{bmatrix}.$$

Let $\det(A) = \Delta$.

(a) [1pt] Find $\det(B)$. Provide your reasons.

$A \xrightarrow{\rho_1 \leftrightarrow \rho_2} B$, so $\det(B) = -\det(A) = \underline{\underline{-\Delta}}$

(b) [1pt] Find $\det(C)$. Provide your reasons.

$A \xrightarrow{\rho_1 : \times 2} \xrightarrow{\rho_2 : \times 3} \xrightarrow{\rho_3 : \times 4} C$, so $\det(C) = 2 \cdot 3 \cdot 4 \det(A)$

$= \underline{\underline{24\Delta}}$.

(c) [1pt] Find $\det(D)$. Provide your reasons.

$A \xrightarrow[\rho_3 : +4\rho_1]{\rho_2 : +3\rho_1} D$, so $\det(D) = \det(A) = \underline{\underline{\Delta}}$

(d) [1pt] Find $\det(E)$. Provide your reasons.

$E = \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}}_{M} A$, so $\det(E) = \det(M)\det(A) = \cancel{8\Delta}$ $\underline{9\Delta}$

$\det(M) = 1 + 8$

(e) [1pt] Find $\det(F)$. Provide your reasons.

$\det(F) = 0$ since F contains repeated rows.

2. Let

$$A = \begin{bmatrix} a & b & c & d \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}.$$

(a) [4pt] Find $\det(A)$ in terms of variables a, b, c, and d.

By Laplace expansion,

$$\det(A) = a \cdot \underbrace{\det\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{4+2-2} - b \cdot \underbrace{\det\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{2+2-1-1} + c \cdot \underbrace{\det\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}}_{4-2-2} - d \cdot \underbrace{\det\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{bmatrix}}_{8+2-2-4}$$

$$= 4\cancel{2}a \cancel{+2b} - 2b + 0c - 4d$$

(b) [1pt] Find some **nonzero** values of a, b, c and d such that $\det(A)$ is zero.

Find any nonzero values so that $2a - 2b + 0c - 4d = 0$.

For example, $a = 2$
$b = 2$
$c = 1$
$d = 1$

3. [5pt] Let

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & 12 \\ 12 & -9 & 28 \end{bmatrix}.$$

Write A as a product of elementary matrices.

$$A \xrightarrow[\rho_3 : -12\rho_1]{\rho_2 : -5\rho_1} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 3 & \cancel{-15}\ -8 \end{bmatrix} \xrightarrow[\rho_3 : -3\rho_2]{\rho_1 : +\rho_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\rho_2 : +3\rho_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\rho_3 : +12\rho_1$

$\rho_2 : +5\rho_1$

$\rho_3 : +3\rho_2$

$\rho_1 : -\rho_2$

$\rho_2 : -3\rho_3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 12 & \cancel{12}\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & \cancel{3}\ 0 & 1 \end{bmatrix}$$

4. [5pt] Mathematical essay: Write a few paragraphs to introduce *permutation expansion.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

5. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

be a 10×10 matrix. Find $\det(A)$.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	