國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

第一次期中考 March 20, 2023 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________________

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| Lecturer: | Jephian Lin 林晉宏 |
| Contents: | cover page, **5 pages** of questions, score page at the end |
| To be answered: | on the test paper |
| Duration: | **110 minutes** |
| Total points: | **20 points** + 2 extra points |

**Do not open this packet until instructed to do so.**

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Write down the $3 \times 3$ elementary matrix for the row operation $\rho_3 : +5\rho_2$ and find its determinant.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \qquad \det(E) = 1$$

2. [1pt] Write down the $3 \times 3$ elementary matrix for the row operation $\rho_2 \leftrightarrow \rho_3$ and find its determinant.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \det(E) = -1$$

3. [1pt] Write down the $3 \times 3$ elementary matrix for the row operation $\rho_1 : \times 3$ and find its determinant.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \det(E) = 3$$

4. [2pt] Find the adjugate of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The $ij$-entry of $A^{\text{cof}}$ = $ij$-cofactor of $A = (-1)^{i+j} \cdot \det(A(i|j))$

$$A^{\text{cof}} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then $A^{\text{adj}} = A^{\text{cof}} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

5. [2pt] Find the determinant of

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

$\rho_i = \rho_i - \rho_1$ for $i = 2 \sim 6$.

$\det(A) = \det\left(\begin{bmatrix} 7 & \cdots & 7 \\ & 2 & & 1 \\ & 1 & \ddots & \\ & & & 2 \end{bmatrix}\right) = 7 \cdot \det\left(\begin{bmatrix} 1 & \cdots & 1 \\ & 2 & & 1 \\ & 1 & \ddots & \\ & & & 2 \end{bmatrix}\right) = 7 \cdot \det\left(\begin{bmatrix} 1 & \cdots & 1 \\ & 1 & & 0 \\ & 0 & \ddots & \\ & & & 1 \end{bmatrix}\right) = \underline{\underline{7}}.$

add $2\sim 6$ rows to row 1.

extract 7 from row 1

6. [3pt] Find the determinant of

$$L = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}.$$

Let $L_n = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}$ $n \times n$ mtx. Then $L = L_7$.

Then $\det(L_1) = -2$

$\det(L_2) = \det\left(\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\right) = 3$

By Laplace expansion along 1st row

$\det(L_n) = -2 \cdot \det(L_{n-1}) - \det\begin{bmatrix} 1 & 1 & & 0 \\ 0 & -2 & \ddots & \\ \vdots & 1 & \ddots & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ expand again.

$= -2 \cdot \det(L_{n-1}) - \det(L_{n-2})$

So $\det(L_3) = -2\det(L_2) - \det(L_1) = -2 \cdot 3 - (-2) = -4$

$\det(L_4) = -2(-4) - 3 = 5 \quad \cdots \quad \det(L) = \det(L_7) = \underline{\underline{-8}}$

7. Let

$$A_x = \begin{bmatrix} -x & 1 & 1 & 1 & 1 \\ 1 & -x & 0 & 0 & 0 \\ 1 & 0 & -x & 0 & 0 \\ 1 & 0 & 0 & -x & 0 \\ 1 & 0 & 0 & 0 & -x \end{bmatrix}.$$

(a) [2pt] Find $\det(A_x)$.

Expand 1st row:

$$\det(A_x) = (-x)\cdot \det\begin{bmatrix} -x & & 0 \\ & \ddots & \\ 0 & & -x \end{bmatrix} - \det\begin{bmatrix} 1 & & & \\ \vdots & -x & & \\ & & -x & \\ 1 & & & -x \end{bmatrix} + \det\begin{bmatrix} 1 & -x & & \\ \vdots & & -x & \\ 1 & & & -x \end{bmatrix} - \det\begin{bmatrix} 1 & -x & & \\ \vdots & & -x & \\ 1 & & & -x \end{bmatrix} + \det\begin{bmatrix} 1 & -x & & \\ \vdots & & -x & \\ 1 & & & -x \end{bmatrix}$$

$$= (-x)\cdot(-x)^4 - (-x)^3 \times 4$$

$$= -x^5 + 4x^3.$$

(b) [3pt] Find all $x$ such that $\det(A_x) = 0$. For each of such $x$, find a nonzero vector $\mathbf{v}$ in $\ker(A_x)$.

Since $\det(A_x) = -x^5 + 4x^3 = -x^3(x^2-4)$,

$\det(A_x) = 0$ when $x = 0, 2, -2$.

① $x=0$. Use row operations to find $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \in \ker(A_0)$.

② $x=2$, By row operations, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \ker(A_2)$

③ $x=-2$, By row operation, $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \in \ker(A_{-2})$.

8. [5pt] Let

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}.$$

Let $E$ be a $2 \times 2$ elementary matrix. Discuss how $E$ changes the shape of $S$ into $ES = \{E\mathbf{v} : \mathbf{v} \in S\}$ and calculate its area. Make sure you consider each of the three types of elementary matrices and give some concrete examples.

9. [extra 2pt] Let

$$A = \begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{y} & - \\ - & \mathbf{z} & - \end{bmatrix} \text{ and } B = \begin{bmatrix} - & \mathbf{x}+\mathbf{y} & - \\ - & \mathbf{y}+2\mathbf{z} & - \\ - & \mathbf{z}+3\mathbf{x} & - \end{bmatrix}$$

be $3 \times 3$ matrices. Suppose $\det(A) = 1$. Find $\det(B)$.

$$B = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}}_{E} \cdot A.$$

So $\det(B) = \det(E) \cdot \det(A)$

$= 7 \cdot \det(A).$

$= 7.$

[END]

| Page | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 2 | |
| Total | 20 (+2) | |