國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104A / GEAI 1209A: Linear Algebra II**

期末考 May 29, 2023 **Final Exam**

姓名 Name : ____________________

學號 Student ID # : solution

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Find a real matrix that is not diagonalizable. Provide your reasons.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad a.m(0) = 2 \text{ but } g.m(0) = 1.$$

2. [1pt] Find a real matrix that is diagonalizable but some of its eigenvalue is not real. Provide your reasons.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{char poly } p_A(x) = x^2 + 1$$
$$\Rightarrow \text{spec}(A) = \{i, -i\}.$$

3. [1pt] Find a real matrix that is diagonalizable but its eigenbasis is not mutually orthogonal. Provide your reasons.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{spec}(A) = \{0, 1\}.$$
$$\text{eigvecs} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

4. [2pt] Find the spectral decomposition of

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\text{charpoly } p(x) = \overset{x^2-4x-5}{(x-5)(x+1)}$$

$$\lambda_1 = 5 \to \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2}$$

$$\lambda_2 = -1 \to \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{2}$$

$$\text{So } A = 5 \cdot \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + (-1) \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$-18+2+6+12=2$
$-45+6+15+28=4$
$-63+8+21+40=6$
$-81+10+27+52=8.$

5. [5pt] Let

$$A = \begin{bmatrix} -18 & 1 & 2 & 3 \\ -45 & 3 & 5 & 7 \\ -63 & 4 & 7 & 10 \\ -81 & 5 & 9 & 13 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

It is known that $\mathbf{v}$ is an eigenvector of A. Find the spectrum of A.

Let $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix}$. Then $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}$.

Compute $Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 3 \\ 4 & 3 & 5 & 7 \\ 6 & 4 & 7 & 10 \\ 8 & 5 & 9 & 13 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Since $\operatorname{spec}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \{0, 0, 3\}$.

$$\underline{\operatorname{Spec}(A) = \{2, 0, 0, 3\}}.$$

6. [5pt] Let

$$A = \begin{bmatrix} \overset{4}{\cancel{3}} & 1 & 1 & 1 & 1 \\ 1 & \overset{4}{\cancel{3}} & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

Find the spectrum of A. You may also give some partial information of the spectrum of A to get some partial credits, e.g., $\lambda_1 \geq 10$.

① $\lambda = 2$

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{null} \geq 2.$$

② $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix} \longrightarrow \text{null} \geq 1.$$

$\Rightarrow$ only 2 eigvals missing.

Equitable partition

$$A = \left[\begin{array}{cc|ccc} 4 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 \\ \hline 1 & 1 & 2 & & \\ 1 & 1 & & 2 & \\ 1 & 1 & & & 2 \end{array}\right] \longrightarrow$$

quotient mtx

$$B = \begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix}$$

$\to$ charpoly $p_B(x) = x^2 - 7x + 4$

$$\text{root} = \frac{7 \pm \sqrt{49 - 16}}{2} = \frac{7 \pm \sqrt{33}}{2}$$

So $\text{spec}(A) = \left\{ 2, 2, 3, \frac{7 \pm \sqrt{33}}{2} \right\}$.

7. Let A be an $m \times n$ matrix and $A = U\Sigma V^\top$ its singular value decomposition. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of the *singular value decomposition* as clear as possible.

   (a) [2pt] Describe the shape and the properties of matrices U, Σ and V. For example, M is an $m \times m$ diagonal matrix.

   (b) [2pt] Let $\mathbf{u}_i$'s and $\mathbf{v}_i$'s be the columns of U and V, respectively. Describe their relations.

   (c) [1pt] Give some reasons about why the singular value decomposition is important.

8. [extra 5pt] Let A be the 9×9 matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Find the inertia of A.

Apply the operations

$$\begin{cases} \rho_2 : +(-\rho_1) \\ \kappa_2 : +(-\kappa_1) \end{cases}$$

$$\begin{cases} \rho_3 : +\rho_2 \\ \kappa_3 : +\kappa_2 \end{cases}$$

$\vdots$

and get

$$\begin{bmatrix} 1 & & & & & & & & \\ & -1 & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & 1 & & & & \\ & & & & & -1 & & & \\ & & & & & & 1 & & \\ & & & & & & & -1 & \\ & & & & & & & & 0 \end{bmatrix}.$$

$\Rightarrow \text{Iner}(A) = (4, 4, 1)$

9. [extra 2pt] Let

$$f(x, y) = x^2 + 4xy + 4y^2.$$

Find the maximum value of $f(x, y)$ subject to $x^2 + y^2 = 1$.

$$f(x,y) = [x\ \ y]\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$$

So $\max\limits_{x^2+y^2=1} f(x,y) = \max\limits_{x^2+y^2=1} R_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \text{max eigval of } A$

Since $\operatorname{spec}(A) = \{0, \cancel{2}\ 5\}$ $\quad 0 \to \begin{bmatrix} 2 \\ -1 \end{bmatrix}/\sqrt{5}$, $\quad 5 \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}/\sqrt{5}$

max value $= 5$.

when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	