國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

期末考 December 18, 2023 **Final Exam**

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學號 Student ID # :

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

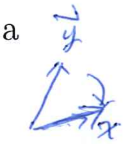
Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto span $\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right)$. Is f a bijection? **Provide your reasons.**

No, f is not surjective, e.g. $\begin{bmatrix}1\\0\end{bmatrix}$ is not in the range

2. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by 60°. Is f a bijection? **Provide your reasons.**



Yes. For any $\vec{y} \in \mathbb{R}^2$, choose $\vec{x}$ as the vector obtained from $\vec{y}$ ~~be~~ by clockwise rotation by $60°$, then $\vec{x}$ is the only vector with $f(\vec{x}) = \vec{y}$.

3. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}x+y\\x+y\end{bmatrix}.$$

Find range(f).

$\text{range}(f) = \left\{ \begin{bmatrix}x+y\\x+y\end{bmatrix} : x, y \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix}1\\1\end{bmatrix} \right\}$.

4. [1pt] Let $\mathcal{P}_d$ be the vector space of polynomials of degree at most d. Let $f : \mathcal{P}_2 \to \mathcal{P}_3$ be the function defined by $p \mapsto x \cdot p - p'$, where p' is the derivative of the polynomial p. Is f a linear function? **Provide your reasons.**

Yes. Note that $(\cancel{f}_h + g)' = \cancel{f}_h' + g'$ and $(r\cdot \cancel{f}_h)' = r\cdot \cancel{f}_h'$ for any functions $\cancel{f}_h, g$ and value r.

So $f(p_1+p_2) = x\cdot(p_1+p_2) - (p_1+p_2)' = (x\cdot p_1 + p_1') + (x\cdot p_2 + p_2') = f(p_1) + f(p_2)$.

$f(r\cdot p) = x\cdot \cancel{p}(rp) - (rp)' = r\cdot xp - rp' = r\cdot f(p)$.

5. [1pt] Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $x \mapsto x^3$. Is f a linear function? **Provide your reasons.**

No. For example, $f(2x) = 8x^3 \neq 2\cdot f(x)$.

6. Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

Let $\alpha = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases of

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}.$$

(a) [1pt] Find the vector $\mathbf{u}$ such that $[\mathbf{u}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$[\vec{u}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ means $\vec{u} = 1 \cdot \vec{b}_1 + 1\vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \underline{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}$.

(b) [1pt] Find $[\mathbf{v}]_\alpha$.

By direct calculation, $\vec{v} = -\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, so $\underline{[\vec{v}]_\alpha = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}}$.

(c) [1pt] Find $[\mathbf{v}]_\beta$.

By direct calculation, $\vec{v} = 1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, so $\underline{[\vec{v}]_\beta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$.

(d) [2pt] Find the change of basis matrix $[\mathrm{id}]_\alpha^\beta$ from α to β.

$$[\mathrm{id}]_\alpha^\beta = \begin{bmatrix} [\vec{a}_1]_\beta & [\vec{a}_2]_\beta \end{bmatrix} = \underline{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}.$$

7. Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ be a function defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - 5x_2 - 3x_3 - 2x_4 + 9x_5 \\ x_4 - 4x_5 \\ 5x_1 - 25x_2 - 15x_3 - 13x_4 + 57x_5 \end{bmatrix}$$

(a) [2pt] Find a matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^5$.

$$A = [f] = \begin{bmatrix} | & & | \\ f(\vec{e}_1) & \cdots & f(\vec{e}_5) \\ | & & | \end{bmatrix} = \underline{\begin{bmatrix} 1 & -5 & -3 & -2 & 9 \\ 0 & 0 & 0 & 1 & -4 \\ 5 & -25 & -15 & -13 & 57 \end{bmatrix}}$$

(b) [3pt] Find a basis of $\ker(f)$.

Run the row operations,

$$A \rightsquigarrow \begin{bmatrix} 1 & -5 & -3 & -2 & 9 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & -3 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -5 & -3 & \overset{0}{\cancel{-2}} & \overset{1}{\cancel{9}} \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free variables			leading	
x_2	x_3	x_5	x_1	x_4
1	0	0	5	0
0	1	0	3	0
0	0	1	-1	4.

Set $\longrightarrow$ solve.

$$\Rightarrow \vec{h}_1 = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \vec{h}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \vec{h}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$\underline{\text{basis} = \{\vec{h}_1, \vec{h}_2, \vec{h}_2\}}$

8. [5pt] Mathematical essay: Write a few paragraphs to introduce *isomorphism.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

9. [extra 5pt] Let

$$\alpha = \left\{ \overset{\vec{a}_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \overset{\vec{a}_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \overset{\vec{a}_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \right\} \text{ and } \beta = \left\{ \overset{\vec{b}_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \overset{\vec{b}_2}{\begin{bmatrix}-1\\1\\0\end{bmatrix}}, \overset{\vec{b}_3}{\begin{bmatrix}-1\\0\\1\end{bmatrix}} \right\}$$

be bases of $\mathbb{R}^3$. Given that

$$[f]^{\beta}_{\beta} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

find $[f]^{\alpha}_{\alpha}$.

~~Cact~~ Calculate

$$[id]^{\beta}_{\alpha} = \begin{bmatrix} | & | & | \\ [\vec{a}_1]_\beta & [\vec{a}_2]_\beta & [\vec{a}_3]_\beta \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$[id]^{\alpha}_{\beta} = \begin{bmatrix} | & | & | \\ [\vec{b}_1]_\alpha & [\vec{b}_2]_\alpha & [\vec{b}_3]_\alpha \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$[f]^{\alpha}_{\alpha} = [id]^{\alpha}_{\beta}\, [f]^{\beta}_{\beta}\, [\cancel{f}\ id.]^{\beta}_{\alpha}$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

10. [extra 2pt] Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

be the vector space of all 2×2 matrices. Let

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\beta = \{E_1, E_2, E_3, E_4\}$ is a basis of V. Define a function $f : V \to V$ by

$$X \mapsto \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} X.$$

Find the matrix representation $[f]_\beta^\beta$ of the function f with respect to the bases β and β.

$$f(E_1) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \xmapsto{[\cdot]_\beta} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$f(E_2) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$f(E_3) = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$f(E_4) = \begin{bmatrix} 0 & 3 \\ 0 & 4 \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \end{bmatrix}$$

$$[f]_\beta^\beta = \begin{bmatrix} | & | & | & | \\ [f(E_1)]_\beta & [f(E_2)]_\beta & [f(E_3)]_\beta & [f(E_4)]_\beta \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	