國立中山大學	NATIONAL SUN YAT-SEN UNIVERSITY
線性代數（一）	**MATH 103A / GEAI 1215A: Linear Algebra I**
期末考 December 18, 2023	**Final Exam**

姓名 Name : ____________________

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	6 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto $\text{span}\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}\right)$. Is f a bijection? **Provide your reasons.**

2. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by $60°$. Is f a bijection? **Provide your reasons.**

3. [1pt] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}x+y\\x+y\end{bmatrix}.$$

Find range(f).

4. [1pt] Let $\mathcal{P}_d$ be the vector space of polynomials of degree at most d. Let $f : \mathcal{P}_2 \to \mathcal{P}_3$ be the function defined by $p \mapsto x \cdot p - p'$, where p' is the derivative of the polylnomial p. Is f a linear function? **Provide your reasons.**

5. [1pt] Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $x \mapsto x^3$. Is f a linear function? **Provide your reasons.**

6. Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

Let $\alpha = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases of

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}.$$

(a) [1pt] Find the vector $\mathbf{u}$ such that $[\mathbf{u}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) [1pt] Find $[\mathbf{v}]_\alpha$.

(c) [1pt] Find $[\mathbf{v}]_\beta$.

(d) [2pt] Find the change of basis matrix $[\mathrm{id}]_\alpha^\beta$ from α to β.

7. Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ be a function defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - 5x_2 - 3x_3 - 2x_4 + 9x_5 \\ x_4 - 4x_5 \\ 5x_1 - 25x_2 - 15x_3 - 13x_4 + 57x_5 \end{bmatrix}$$

(a) [2pt] Find a matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^5$.

(b) [3pt] Find a basis of $\ker(f)$.

8. [5pt] Mathematical essay: Write a few paragraphs to introduce *isomorphism.*

   Your score will be based on the following criteria.

   - The definition is clear.
   - Some sentences are added to explain the definition.
   - Examples or pictures are included to help understanding.
   - The sentences are complete.

9. [extra 5pt] Let

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } \beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be bases of $\mathbb{R}^3$. Given that

$$[f]_\beta^\beta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

find $[f]_\alpha^\alpha$.

10. [extra 2pt] Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

be the vector space of all 2×2 matrices. Let

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\beta = \{E_1, E_2, E_3, E_4\}$ is a basis of V. Define a function $f : V \to V$ by

$$X \mapsto \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} X.$$

Find the matrix representation $[f]_\beta^\beta$ of the function f with respect to the bases β and β.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	20 (+7)	