國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

第一次期中考 March 21, 2022 **Midterm 1**

姓名 Name : Solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	5 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] What is the elementary matrix corresponding the row operation $\rho_1 \leftrightarrow \rho_3$ applied on matrices with 3 rows? What is its determinant?

$$E = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \qquad \det(E) = -1.$$

2. [1pt] What is the elementary matrix corresponding the row operation $\rho_3 : \times 3$ applied on matrices with 3 rows? What is its determinant?

$$E = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix} \qquad \det(E) = 3$$

3. [1pt] What is the elementary matrix corresponding the row operation $\rho_1 : +7\rho_3$ applied on matrices with 3 rows? What is its determinant?

$$E = \begin{pmatrix} 1 & & 7 \\ & 1 & \\ & & 1 \end{pmatrix} \qquad \det(E) = 1.$$

4. [2pt] Find a 4×4 matrix A such that $\det(A) = 7$ and every entry of A is nonzero. (Explain why your answer is correct.)

$$+1 \begin{pmatrix} 7 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \xrightarrow[\text{to make entries nonzero}]{\text{use row operations}} \begin{pmatrix} 7 & 1 & 1 & 1 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} +1 \longrightarrow \begin{pmatrix} 7 & 1 & 1 & 1 \\ 7 & 2 & 1 & 1 \\ 7 & 1 & 2 & 1 \\ 7 & 1 & 1 & 2 \end{pmatrix}$$

5. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 5 & 25 & 125 & 625 \\ 1 & 6 & 36 & 216 & 1296 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 0 & x \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 5 & 25 & 125 & 625 \\ 1 & 6 & 36 & 216 & 1296 \end{bmatrix}$$

(a) [1pt] Find $\det(A)$.

Vandermonde mtx

$$\Rightarrow \det(A) = (2-1)(3-2)(5-3)(6-5)(3-1)(5-2)(6-3)(5-1)(6-2)(6-1)$$
$$= 1\cdot1\cdot2\cdot1\cdot2\cdot3\cdot3\cdot4\cdot4\cdot5 = \underline{\underline{2880}}$$

(b) [2pt] Find the last row of A^{-1}.

last row of A^{-1} = last column of $A^{cof} / \det(A)$.

$$\frac{1,5\text{-cofactor}}{\det(A)} = +1 \cdot \frac{(3-2)(5-3)(6-5)(5-2)(6-3)(6-2)}{\det(A)} = (+1)\cancel{(3-1)(}\frac{1}{(2-1)(3-1)(5-1)(6-1)} = +1 \cdot \frac{1}{40}$$

$$\frac{2,5\text{-cofactor}}{\det(A)} = (-1)\cdot\frac{1}{(2-1)(3-2)(5-2)(6-2)} = -\frac{1}{12}$$

$$\frac{3,5\text{-cofactor}}{\det(A)} = 1\cdot\frac{1}{(3-1)(3-2)(5-3)(6-3)} = \frac{1}{12}$$

$$\frac{4,5\text{-cofactor}}{\det(A)} = (-1)\frac{1}{(5-1)(5-2)(5-3)(6-5)} = -\frac{1}{24}$$

$$\frac{5,5\text{-cofactor}}{\det(A)} = 1\quad\frac{1}{(6-1)(6-2)(6-3)(6-5)} = \frac{1}{60}$$

$$\Rightarrow \text{last row} = \left(\frac{1}{40}\ \ -\frac{1}{12}\ \ \frac{1}{12}\ \ -\frac{1}{24}\ \ \frac{1}{60}\right)$$

(c) [2pt] Find the x such that $\det(B) = 0$.

$$\det(B) = \det\underbrace{\begin{pmatrix} 2 & 4 & 8 & 16 \\ 3 & & & \\ 5 & & \cdots & \\ 6 & & & \end{pmatrix}}_{X} + x\underbrace{\begin{pmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & \cdots & \\ 1 & 6 & \cdots & \end{pmatrix}}_{Y} = \det(X) + x\det(Y).$$

Note that $X = \begin{pmatrix} 2 & & & \\ & 3 & & \\ & & 5 & \\ & & & 6 \end{pmatrix} Y \Rightarrow \det(X) = 2\cdot3\cdot5\cdot6\cdot\det(Y)$

若 $\det(B)=0$，則 $x = -\det(X)/\det(Y) = -2\cdot3\cdot5\cdot6 = \underline{\underline{-180}}$

6. Show that $\det(A) = \det(A^\top)$ for any square matrix A.

令 A 為 $n \times n$ 矩陣

① 若 A 不可逆，則 $\operatorname{rank}(A) < n$。

因 $\operatorname{rank}(A^T) = \operatorname{rank}(A) < n$，

故 A^T 不可逆。

此情況下 $\det(A) = \det(A^T)$.

② 若 A 可逆，則 A 可寫成基本矩陣 $E_1, \cdots, E_k$ 的乘積

$$A = E_1 \cdots E_k.$$

Claim：若 E 為基本矩陣，則 $\det(E) = \det(E^T)$.

case 1. $\rho_i \leftrightarrow \rho_j$ 的基本矩陣.

$$E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \Rightarrow E^T = E，故 \det(E) = \det(E^T)$$

case 2. $\rho_i : \times k$ 的.

$$E = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & k & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \Rightarrow E^T = E，故 \det(E) = \det(E^T)$$

case 3. $\rho_i : +k\rho_j$

$$E = \begin{pmatrix} 1 & & k \\ & \ddots & \\ & & 1 \end{pmatrix} \Rightarrow \det(E) = 1 = \det(E^T)$$

~~因此 $\det(A^T) = \det($~~

因 $A^T = E_k^T \cdots E_1^T$

故 $\det(A^T) = \det(E_k^T) \cdot \cdots \cdot \det(E_1^T)$

$= \det(E_1) \cdot \cdots \cdot \det(E_k)$

$= \det(A).$

7. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是行列式值（determinant）。

請敘述行列式值的定義，並解釋定義中每一條規則的直觀意義。請以自己的方式、盡量白話的敘述、或是比喻來說明爲什麼要考慮這樣的概念？請給一些能幫助他人理解的例子，並提出一些這個概念的相關性質；有必要的話可以加上一些圖來輔助說明。格式沒有限制，篇幅大約半面到一面。

(If Chinese is not your native language, you may use English or the language that you prefer.)

8. [extra 2pt] Let A be the 9×9 matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Find $\det(A)$.

有用的 $\sigma \in S_9$ 只有 2個

σ	sgn.	ω	sgn. ω
(123456789)	$(-1)^{9+1} = 1$	1	1
(198765432)	$(-1)^{9+1} = 1$	1	1 (+
			2.

$\Rightarrow \det(A) = 2$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	