國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103A / GEAI 1215A: Linear Algebra I**

期末考 December 19, 2022 **Final Exam**

姓名 Name : solution

學號 Student ID # :

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **6 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using a calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- Please answer the problems in English.

1. [1pt] Let $X = \{a, b, c\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **injective but not surjective**.

$Y = \{2, 3, 4, 5\}$

f : $a \mapsto 2$, $b \mapsto 3$, $c \mapsto 4$, 5

2. [1pt] Let $X = \{a, b, c\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **surjective but not injective**.

$Y = \{2, 3\}$

f : $a \mapsto 2$, $b \mapsto 3$, $c \mapsto 3$

3. [1pt] Let $X = \{a, b, c\}$. Pick a set $Y \subseteq \{1, 2, 3, 4, 5\}$ and define a function $f : X \to Y$ such that f is **bijective**.

$Y = \{3, 4, 5\}$

f : $a \mapsto 3$, $b \mapsto 5$, $c \mapsto 4$

4. [1pt] Define a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that f is **linear**.

Define $f(x, y) = (x, y, 0)$ for all $(x, y) \in \mathbb{R}^2$.

5. [1pt] Define a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that f is **not linear**.

Define $f(x, y) = (x^2, y^2, 0)$ for all $(x, y) \in \mathbb{R}^2$.

6. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Let

$$\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix}3\\2\\1\end{bmatrix}, \text{ and}$$

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix}2\\1\end{bmatrix}.$$

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear function such that $f(\mathbf{u}_i) = \mathbf{v}_i$ for $i = 1, 2, 3$.

(a) [1pt] Find $f(\mathbf{u}_1 + \mathbf{u}_2)$.

$f(\vec{u}_1 + \vec{u}_2) = f(\vec{u}_1) + f(\vec{u}_2) = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix}2\\3\end{bmatrix}$

(b) [2pt] Find $f(\mathbf{e}_2)$ and $f(\mathbf{e}_3)$.

$\vec{e}_2 = \vec{u}_2 - 2\vec{u}_1$

So $f(\vec{e}_2) = f(\vec{u}_2 - 2\vec{u}_1) = \vec{v}_2 - 2\vec{v}_1 = \begin{bmatrix}-1\\-3\end{bmatrix}$

$\vec{e}_3 = \vec{u}_3 - 2\vec{u}_2 + \vec{u}_1$

So $f(\vec{e}_3) = \vec{v}_3 - 2\vec{v}_2 + \vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$.

(c) [1pt] Find the matrix representation $[f]$ such that $[f]\mathbf{u} = f(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^3$.

Note: $f(\vec{e}_1) = f(\vec{u}_1) = \begin{bmatrix}1\\2\end{bmatrix}$, So $[f] = \begin{bmatrix}1 & -1 & 1\\2 & -3 & 1\end{bmatrix}$.

(d) [1pt] Find rank(f) and null(f).

$\begin{bmatrix}1 & -1 & 1\\2 & -3 & 1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & -1 & 1\\0 & -1 & -1\end{bmatrix}$.

rank(f) = # of pivots = 2.

null(f) = 3 − rank(f) = 1.

7. Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Let $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis of $\mathbb{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 5 \\ -12 \end{bmatrix}, \quad \text{and } \mathbf{u}_3 = \begin{bmatrix} -1 \\ 4 \\ -8 \end{bmatrix}.$$

(a) [2pt] Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that

$$[\mathbf{v}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Find $\mathbf{v}$ and $[\mathbf{w}]_\beta$.

Since $[\vec{v}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = 1\vec{u}_1 + 2\vec{u}_2 + 3\vec{u}_3 = \begin{bmatrix} -6 \\ 20 \\ -43 \end{bmatrix}$

~~Solve $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{w}$. by~~

$$[\vec{w}]_\beta = [id]_\alpha^\beta \vec{w} = \begin{bmatrix} 8 & -4 & -3 \\ 4 & -3 & -2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ -8 \\ 6 \end{bmatrix}$$

(b) [3pt] Find the change of basis matrices $[\mathrm{id}]_\alpha^\beta$ and $[\mathrm{id}]_\beta^\alpha$.

$$[id]_\beta^\alpha = \begin{bmatrix} | & | & | \\ [\vec{u}_1]_\alpha & [\vec{u}_2]_\alpha & [\vec{u}_3]_\alpha \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 4 \\ 5 & -12 & -8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -2 & 5 & 4 & 0 & 1 & 0 \\ 5 & -12 & -8 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & -2 & -3 & -5 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & 2 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -4 & -3 \\ 0 & 1 & 0 & 4 & -3 & -2 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right]$$

$$[id]_\alpha^\beta = \left([id]_\beta^\alpha\right)^{-1} = \begin{bmatrix} 8 & -4 & -3 \\ 4 & -3 & -2 \\ -1 & 2 & 1 \end{bmatrix}$$

8. Let V be a vector space and $\beta = \{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ a basis of V. Suppose you are talking to people who have never learned linear algebra. Follow the guidelines below and try to explain the concept of the *vector representation* as clear as possible.

   (a) [2pt] Suppose $\mathbf{v} \in V$ is a vector. Define what is the vector representation $[\mathbf{v}]_\beta$ with respect to β and use a few sentences to explain the definition.

   (b) [1pt] What might happen if β is not a basis?

   (c) [2pt] Provide an example of $[\mathbf{v}]_\beta$ with $V = \mathbb{R}^2$ and an example of $[\mathbf{v}]_\beta$ with $V = \mathcal{P}_1$, the space of all real polynomials of degree at most 1.

9. [extra 5pt] Let $\mathcal{M}_{2,3}$ be the space of all 2×3 real matrices. Let

$$\beta = \{E_{1,1}, E_{1,2}, E_{1,3}, E_{2,1}, E_{2,2}, E_{2,3}\}$$

be the standard basis of $\mathcal{M}_{2,3}$, where

$$E_{1,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_{2,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, E_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the linear function $f : \mathcal{M}_{2,3} \to \mathcal{M}_{2,3}$ defined by $f(X) = AX$ with

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find the matrix representation $[f]_{\beta}^{\beta}$.

See ver. A.

10. [extra 2pt] Let $\mathcal{P}_3$ be the space of all real polynomials of degree at most 3. Let

$$\beta = \{f_1(x), f_2(x), f_3(x), f_4(x)\}$$

be a basis of $\mathcal{P}_3$, where

$$f_1(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)},$$
$$f_2(x) = \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)},$$
$$f_3(x) = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)},$$
$$f_4(x) = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}.$$

Let $p(x) = 1 - 2x + 3x^2 - 4x^3$. Find the vector representation $[p(x)]_\beta$.

See ver. A.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	(+5)	
6	(+2)	
Total	20 (+7)	