國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） MATH 103 / GEAI 1215: Linear Algebra I

第一次期中考 November 1, 2021 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **5 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**20 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \end{bmatrix} \text{ and } \mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) [1pt] Find a vector in Row(A) that is nowhere zero （每一項都不是零）.

$\underline{(1,2,1,1,1)} = (1,2,0,0,0) + (0,0,1,1,1) \in Row(A)$.

(b) [1pt] Find a vector in Col(A) that is nowhere zero.

$\cancel{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ $\underline{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \in Col(A)$

(c) [1pt] Find a vector in ker(A) that is nowhere zero.

Take $\vec{x} = \underline{\begin{pmatrix} -2 \\ 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}}$. Then $A\vec{x} = \vec{0}$, so $\vec{x} \in \ker(A)$.

(d) [1pt] Find a vector in $\mathbf{p} + \ker(A)$ that is nowhere zero.

Take $\vec{p} + \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -2 \\ 1 \\ 1 \end{pmatrix} = \underline{\begin{pmatrix} -2 \\ 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}} \in \vec{p} + \ker(A)$

(e) [1pt] Let

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find a 3×3 matrix E such that $EA = B$.

$A \xrightarrow{\rho_3 :+ (-2)\rho_2} B$. We may pick E as the elementary matrix of $\rho_3 :+ (-2)\rho_2$, which is $E = \underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}}$.

2. Let

$$A = \begin{bmatrix} 1 & -2 & -2 & -2 \\ 3 & -6 & -6 & -5 \\ -11 & 22 & 22 & 19 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}.$$

(a) [2pt] Find the reduced row echelon form of the augmented matrix $[\, A \,|\, \mathbf{b} \,]$.

$$[A|b] = \left[\begin{array}{cccc|c} 1 & -2 & -2 & -2 & 0 \\ 3 & -6 & -6 & -5 & 1 \\ -11 & 22 & 22 & 19 & -3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -2 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) [3pt] Find $\mathbf{p}$, $\mathbf{h}_1$, $\mathbf{h}_2$ such that

$$\{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{b}\} = \mathbf{p} + \operatorname{span}(\{\mathbf{h}_1, \mathbf{h}_2\}).$$

$$\begin{cases} x_1 - 2x_2 - 2x_3 = 2 \\ x_4 = 1 \end{cases}$$

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令 $x_2 = x_3 = 0$ 解 $A\vec{x} = \vec{b}$ $\Rightarrow$ $x_4 = 1,\ x_1 = 2$ $\Rightarrow \vec{p} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

令 $x_2 = 1, x_3 = 0$ 解 $A\vec{x} = \vec{0}$ $\Rightarrow$ $x_4 = 0,\ x_1 = 2$ $\Rightarrow \vec{h}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

令 $x_2 = 0, x_3 = 1$ 解 $A\vec{x} = \vec{0}$ $\Rightarrow$ $x_4 = 0,\ x_1 = 2$ $\Rightarrow h_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) [3pt] Find $\mathbf{w}$ and $\mathbf{h}$ such that $\mathbf{b} = \mathbf{w} + \mathbf{h}$ with $\mathbf{w} \in \mathrm{Col}(A)$ and $\mathbf{h} \in \mathrm{Col}(A)^{\perp}$.

Let $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$ $\Rightarrow \vec{w}$ = projection of $\vec{b}$ onto $\mathrm{Col}(A)$

Then $\mathrm{Col}(A) = \mathrm{Col}(B)$.

= projection of $\vec{b}$ ~~cor~~ onto $\mathrm{Col}(B)$

$$= B(B^T B)^{-1} B^T \cdot \vec{b}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 10 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}.$$

$$\vec{w} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{h} = \vec{b} - \vec{w} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}$$

(b) [2pt] Let θ be the angle between $\mathbf{b}$ and $\mathbf{w}$. Find $\cos\theta$.

$$\cos\theta = \frac{\langle \vec{b}, \vec{w} \rangle}{\|\vec{b}\| \cdot \|w\|} = \frac{1/2}{1 \cdot \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2}$$

4. [5pt] 數學作文：請寫一篇短文來向沒修過線性代數的朋友介紹什麼是子空間（subspace）。

   請以盡量白話的敘述、或是比喻來介紹什麼是子空間？爲什麼要考慮這樣的概念？並給一些能幫助他人理解的例子（正面的、反面的）；有必要的話可以加上一些圖來輔助說明。格式沒有限制，篇幅大約半面到一面。

   (If Chinese is not your native language, you may use English or the language that you prefer.)

5. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find a 5×5 matrix E such that $E^{\cancel{T}} \cancel{J} E^{T} = I$.
A

$$A \xrightarrow[\text{do the symmetric column operation}]{\rho_2 : +(-1)\rho_1,\ \rho_3 : +(-1)\rho_1,\ \vdots,\ \rho_5 : +(-1)\rho_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow[\text{col oper also}]{\rho_3, \rho_4, \rho_5 : +(-1)\rho_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$E_1 = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & & 1 & & \\ -1 & & & 1 & \\ -1 & & & & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & -1 & 1 & & \\ & -1 & & 1 & \\ & -1 & & & 1 \end{pmatrix}.$$

$$\xrightarrow[\text{col oper}]{\rho_4, \rho_5 : +(-1)\rho_3} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & 1 & 2 \end{pmatrix} \xrightarrow[\text{col oper}]{\rho_5 : +(-1)\rho_4} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & -1 & 1 & \\ & & -1 & & 1 \end{pmatrix} \qquad E_4 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & -1 & 1 \end{pmatrix}$$

$$\Rightarrow E = E_4 E_3 E_2 E_1 = \underline{\begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}}$$

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	2	
Total	20 (+2)	