Math585 Homework ?

Note: There are several options to finish the homework. You may find a definition in the textbook and provide one example that meets the definition and one example that does not. You may find a theorem in the textbook and provide an example that witness the theorem. You may also find an exercise in the textbook and finish it. To submit the k-th homework, you may give me a paper copy or send me the electronic copy. The deadline is announced on the course website.

1. [Definition of positive definite matrix] A $n \times n$ symmetric matrix M is *positive definite* if $\mathbf{x}^\top M\mathbf{x}$ is positive for any $\mathbf{x} \in \mathbb{R}^n$.

   Solution. Let
   $$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
   Then A is positive definite while B is not. According to the definition, a matrix that is not symmetric is not in the consideration of being positive definite or not.

2. [Cauchy–Binet formula] Let A and B be an $m \times n$ matrix and an $n \times m$ matrix with $m \leqslant n$, respectively. Then

$$\det(AB) = \sum_{\alpha} \det(A[:, \alpha]) \det(B[\alpha, :]),$$

where α runs through all m-subsets of $\{1, \ldots, n\}$.

Solution. Let

$$A = B^\top = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

has determinant $\det(AB) = 6$. Let's compute it in a different way. Let α be a 2-subset of $\{1, 2, 3\}$, so α can be one of $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$. When $\alpha = \{1, 2\}$,

$$A[:, \alpha] = B[\alpha, :]^\top = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and $\det(A[:, \alpha]) \det(B[\alpha, :]) = 1$. When $\alpha = \{1, 3\}$,

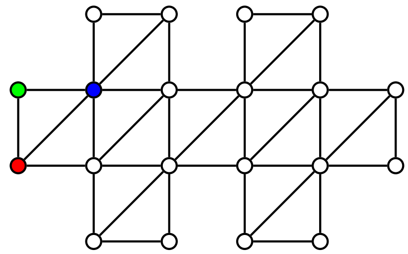
$$A[:, \alpha] = B[\alpha, :]^\top = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

and $\det(A[:, \alpha]) \det(B[\alpha, :]) = 4$. When $\alpha = \{2, 3\}$,

$$A[:, \alpha] = B[\alpha, :]^\top = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

and $\det(A[:, \alpha]) \det(B[\alpha, :]) = 1$. Thus, we get the same answer $\det(AB) = 1 + 4 + 1 = 6$.

3. [Exercise ?.??] Let G be the graph below. Find the chromatic number $k = \chi(G)$ and give a proper k-coloring of G.



Solution. It is an outer-planar graph that contains a triangle, so $\chi(G) = 3$.