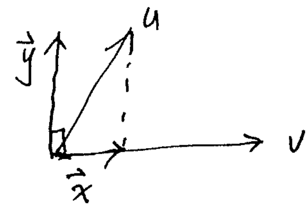
Sample Solution for Sample Questions 5

1. 公式：$\vec{u}$ 在 $\vec{v}$ 上的投影為 $\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v}$.

$\vec{u}\cdot\vec{v} = 1 + (-1) + 1 = 1$

$|\vec{v}|^2 = 1^2 + 1^2 + 1^2 = 3.$



$$\Rightarrow \quad \vec{x} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v} = \frac{1}{3}\vec{v} \quad (\text{投影})$$

$$\vec{y} = \vec{u} - \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -4/3 \\ 2/3 \end{pmatrix}$$

[You may check that $\langle \vec{v}, \vec{y} \rangle = 0$.]

2. Suppose $A = [a_{ij}]$, $B = [b_{ij}]$.

Then

$$i,j\text{-entry of } (AB)^T = j,i\text{-entry of } AB = \sum_{k=1}^{n} a_{jk} b_{ki}$$

$$i,j\text{-entry of } B^T A^T = \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} = \sum_{k=1}^{n} b_{ki}\, a_{kj}$$

$\rightarrow$ the same.

3.

$$Ax = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$$

$$\Rightarrow \langle Ax, y\rangle = [3\ 2\ 1]\begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = -12 .$$

$$A^Ty = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 20 \end{pmatrix}$$

$$\Rightarrow \langle x, A^Ty\rangle = [14\ \ 20]\begin{bmatrix} 2 \\ -2 \end{bmatrix} = -12$$

Thus $\langle Ax, y\rangle = -12 = \langle x, A^Ty\rangle$.

4. Claim: $A\vec{x} = \vec{0} \Leftrightarrow A^TA\vec{x} = 0$.

"$\Rightarrow$" Suppose $A\vec{x} = \vec{0}$

Then $A^TA\vec{x} = A^T(A\vec{x})$

$$= A^T\vec{0} = \vec{0}.$$

長度不同的零向量.

"$\Leftarrow$" Suppose $A^TA\vec{x} = \vec{0}$.

Then $\langle \vec{x}, A^TA\vec{x}\rangle = \langle \vec{x}, \vec{0}\rangle = 0$

$\parallel$

$\Rightarrow \langle A\vec{x}, A\vec{x}\rangle = 0.$

$\Rightarrow |A\vec{x}|^2 = 0$

$\Rightarrow A\vec{x} = \vec{0}.$

注意: A^T 不一定是方陣
也不一定有反矩陣, 所以
不用說

$$A^{T^{-1}}A^TA\vec{x} = A^{T^{-1}}\vec{0}$$
$$\Downarrow$$
$$A\vec{x} = \vec{0}$$

5. Let A be an $m \times n$ matrix. Then A^TA is $n \times n$.

Claim: A has full column rank $\Leftrightarrow$ A^TA is invertible.

Recall: ① A has full column rank

$\Leftrightarrow$ no free variable

$\Leftrightarrow$ $\vec{x} = \vec{0}$ is the only solution of $A\vec{x} = \vec{0}$.

② A^TA is invertible

$\Leftrightarrow$ A^TA is nonsingular

$\Leftrightarrow$ $\vec{x} = \vec{0}$ is the only solution of $A^TA\vec{x} = \vec{0}$.

Claim:

$\vec{x} = \vec{0}$ is the only solution of $A\vec{x} = \vec{0}$

$\Leftrightarrow$ $\vec{x} = \vec{0}$ is the only solution of $A^TA\vec{x} = \vec{0}$.

pf. "$\Rightarrow$" Suppose $A^TA\vec{x} = \vec{0}$

$\Rightarrow A\vec{x} = \vec{0}$ [By Problem 4]

$\Rightarrow \vec{x} = \vec{0}$ [By assumption].

"$\Leftarrow$" Suppose $A\vec{x} = \vec{0}$

$\Rightarrow A^TA\vec{x} = \vec{0}$ [By Problem 4]

$\Rightarrow \vec{x} = \vec{0}$ [By assumption].

6.

$$\langle x, y\rangle = \overline{(3+4i)}\cdot(3+2i) + \overline{(-4i)}(2-3i) + \overline{(2-i)}\cdot i$$

$$= (3-4i)(3+2i) + (4i)(2-3i) + (2+i)\cdot i$$

$$= 17-6i + \overset{12}{\cancel{8}} + \overset{8i}{\cancel{4i}} - 1 + 2i$$

$$= \cancel{22}\, 28 + 4i.$$

$$\langle y, x\rangle = \overline{(3+2i)}(3+4i) + \overline{(2-3i)}(-4i) + \overline{i}\cdot(2-i)$$

$$= (3-2i)(3+4i) + (2+3i)(-4i) + (-i)(2-i)$$

$$= 17+6i + 12 + 8i - 1 - 2i$$

$$= 28 - 4i \quad \leftarrow \quad \overline{\langle x, y\rangle}$$

$$|x| = \sqrt{\langle x, x\rangle}$$

$$= \sqrt{\overline{(3+2i)}(3+2i) + \overline{(2-3i)}(2-3i) + \overline{(i)}(i)}$$

$$= \sqrt{3^2+2^2+2^2+3^2+1^2}$$

$$= \sqrt{27}.$$

7. Since $f(x)$ passes through the 5 points,

$$f(p_1) = a + bp_1 + cp_1^2 + dp_1^3 + ep_1^4 = q_1$$

$$\vdots$$

$$f(p_5) = a + bp_5 + cp_5^2 + dp_5^3 + ep_5^4 = q_5$$

$$\Updownarrow$$

$$\begin{pmatrix} 1 & p_1 & p_1^2 & p_1^3 & p_1^4 \\ 1 & p_2 & p_2^2 & p_2^3 & p_2^4 \\ 1 & p_3 & p_3^2 & p_3^3 & p_3^4 \\ 1 & p_4 & p_4^2 & p_4^3 & p_4^4 \\ 1 & p_5 & p_5^2 & p_5^3 & p_5^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}$$