

Sample Solutions for Sample Questions 3

1.

$$f(E_{1,1}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{Rep}_B} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$f(E_{1,2}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{Rep}_B} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$f(E_{2,1}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{Rep}_B} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$f(E_{2,2}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{Rep}_B} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$S_0 \text{Rep}_{B,B}(f) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$2. \quad f(E_{1,1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T = E_{1,1} \xrightarrow{\text{Rep}_B} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f(E_{1,2}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T = E_{2,1} \xrightarrow{\text{Rep}_B} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$f(E_{2,1}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T = E_{1,2} \xrightarrow{\text{Rep}_B} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$f(E_{2,2}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^T = E_{2,2} \xrightarrow{\text{Rep}_B} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$S_0 \quad \text{Rep}_{B-B}(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.

① Compute $\text{Rep}_{\mathcal{B}_2, \mathcal{B}_1}(f)$

$$f(1) = 0 \xrightarrow{\text{Rep}_{\mathcal{B}_1}} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$f(x) = 1 \xrightarrow{\text{Rep}_{\mathcal{B}_1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(x^2) = 2x \xrightarrow{\text{Rep}_{\mathcal{B}_1}} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\Rightarrow \text{Rep}_{\mathcal{B}_2, \mathcal{B}_1}(f) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

② Compute $\text{Rep}_{\mathcal{B}_1, \mathcal{B}_2}(g)$

$$g(1) = x \xrightarrow{\text{Rep}_{\mathcal{B}_2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$g(x) = \frac{1}{2}x^2 \xrightarrow{\text{Rep}_{\mathcal{B}_2}} \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$$

$$\Rightarrow \text{Rep}_{\mathcal{B}_1, \mathcal{B}_2}(g) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

③

$$\text{Rep}_{\mathcal{B}_2, \mathcal{B}_2}(g \circ f) = \text{Rep}_{\mathcal{B}_1, \mathcal{B}_2}(g) \cdot \text{Rep}_{\mathcal{B}_2, \mathcal{B}_1}(f)$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4.

$$f(1) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{Rep}_{S_3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$f(x) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \xrightarrow{\text{Rep}_{S_3}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$f(x^2) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix} \xrightarrow{\text{Rep}_{S_3}} \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$$

$$\Rightarrow \text{Rep}_{\mathcal{B}_2, S_3}(f) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

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5. Expand $p_1(x)$, $p_2(x)$, $p_3(x)$ first.

$$p_1(x) = \frac{1}{2}(x^2 - 5x + 6)$$

$$p_2(x) = -1(x^2 - 4x + 3)$$

$$p_3(x) = \frac{1}{2}(x^2 - 3x + 2)$$

Compute $\text{Rep}_{S_3, B_2}(g)$

$$g \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = p_1(x) \xrightarrow{\text{Rep}_{B_2}} \begin{pmatrix} 3 \\ -5/2 \\ 1/2 \end{pmatrix} \quad \begin{array}{l} | \\ x \\ x^2 \end{array} \quad \begin{array}{l} \text{Be aware of} \\ \text{the order.} \end{array}$$

$$g \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = p_2(x) \xrightarrow{\text{Rep}_{B_2}} \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$$

$$g \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = p_3(x) \xrightarrow{\text{Rep}_{B_2}} \begin{pmatrix} 1 \\ -3/2 \\ 1/2 \end{pmatrix}$$

$$\Rightarrow \text{Rep}_{S_3, B_2}(g) = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$$

You may check $MN = NM = I_3$.

6.

$$f(\vec{v}_i) = \lambda_i \vec{v}_i \xrightarrow{\text{Rep}_B} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th entry.}$$

$$\Rightarrow \text{Rep}_{B,B}(f) = \begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \leftarrow \text{diagonal matrix}$$

7. $f(\vec{v}_i) = \lambda \vec{v}_i \xrightarrow{\text{Rep}_B} \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

For $i \geq 2$,

$$f(\vec{v}_i) = \lambda \vec{v}_i + \vec{v}_{i-1} \xrightarrow{\text{Rep}_B} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \begin{matrix} (i-1)\text{-th entry} \\ i\text{-th entry} \end{matrix}$$

$$\Rightarrow \text{Rep}_{B,B}(f) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix}$$

\leftarrow Called a Jordan block.