Sample Solutions for Sample Questions 14.

1. $I_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$

$A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$

$A^4 = O.$

Suppose $a_0 I_4 + a_1 A + a_2 A^2 + a_3 A^3 = O$

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ & a_0 & a_1 & a_2 \\ & & a_0 & a_1 \\ & & & a_0 \end{pmatrix} = O \Rightarrow a_0 = a_1 = a_2 = a_3 = 0.$$

$\Rightarrow \{I_4, A, A^2, A^3\}$ is linearly independent.

But $A^4 = O$.

$\Rightarrow p(A) \neq 0$ if $\deg p(x) \leq 3$.

$p(A) = 0$ if $p(x) = x^4$. $\Rightarrow$ min poly is $p(x) = x^4$.

Similarly, the ~~not~~ min poly of $A + \lambda I_4$

is $p(x) = (x - \lambda)^4$.

2.

$$I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 4 & & & \\ & 4 & & \\ & & 9 & \\ & & & 9 \end{pmatrix}$$

Solve $\quad a_0 I + a_1 A + A^2 = 0.$

$$\begin{pmatrix} a_0+2a_1+4 & & & \\ & a_0+2a_1+4 & & \\ & & a_0+3a_1+9 & \\ & & & a_0+3a_1+9 \end{pmatrix} = 0.$$

$$\Rightarrow \begin{cases} a_0+2a_1 = -4 \\ a_0+3a_1 = -9 \end{cases} \Rightarrow \begin{cases} a_0 = 6 \\ a_1 = -5 \end{cases}$$

So $p(A)=0$ when $p(x) = x^2-5x+6 = (x-2)(x-3)$.

(You may check $p(A) \neq 0$ if $\deg p(x) \leq 1$.)

So min poly is $p(x) = x^2-5x+6$.

3.

$$f' = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \det(f') = r\cos^2\theta + r\sin^2\theta = r.$$

4.

$$f' = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi \, \cancel{\cos} \sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix}$$

$$\Rightarrow \det(f') = -r\sin\phi \cdot \det\begin{pmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta \end{pmatrix}$$

$$+ \cos\phi \cdot \det\begin{pmatrix} -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ r\sin\phi\cos\theta & r\cos\phi\sin\theta \end{pmatrix}$$

$$= -r^2(\sin\phi)^3 \cdot \det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} + r^2(\cos\phi)^2\sin\phi \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

$$= -r^2\sin\phi^3 - r^2(\cos\phi)^2\sin\phi$$

$$= -r^2\sin\phi\,(\sin^2\phi + \cos^2\phi) = -r^2\sin\phi.$$

5. $f(\vec{v}) = A\vec{v}$.

That is $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\Rightarrow \begin{cases} y_1 = 1x_1 + 2x_2 + 3x_3 \\ y_2 = 4x_1 + 5x_2 + 6x_3 \end{cases}$$

$$f' = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

So $f' = A$.

In fact, if $f(\vec{v}) = A\vec{v}$,

then $f' = A$.

6.

Since $f(x)$ is over $\mathbb{C}$,

by the Fundamental Theorem of Algebra,

we may write

$$f(x) = (x-\lambda_1) \cdots (x-\lambda_n)$$

($\lambda_1, \ldots, \lambda_n$ might repeat).

① Suppose $\lambda_1, \ldots, \lambda_n$ ~~has~~ contains a multiple root.

Without loss of generality, assume $\lambda_1 = \lambda_2$.

Then $f'(x) = \dfrac{f(x)}{(x-\lambda_1)} + \dfrac{f(x)}{(x-\lambda_2)} + \cdots + \dfrac{f(x)}{x-\lambda_n}$

Note that $\cancel{\dfrac{f(x)}{x-\lambda_1}}$ $\dfrac{f(x)}{x-\lambda_1}$ contains the factor $x-\lambda_2$

$\dfrac{f(x)}{x-\lambda_2}$ — — — — — $x-\lambda_1$

and $\dfrac{f(x)}{x-\lambda_i}$ has factors $x-\lambda_1$ and $x-\lambda_2$ for $i=3,\ldots,n$.

$\Rightarrow$ So $\cancel{f'(\lambda_1)}$ $\lambda_1=\lambda_2$ is a common root for $f(x)$ and $f'(x)$.

② Suppose $\lambda_1, \ldots, \lambda_n$ are all distinct.

Then $f'(x) = \dfrac{f(x)}{x-\lambda_1} + \dfrac{f(x)}{x-\lambda_2} + \cdots + \dfrac{f(x)}{x-\lambda_n}$.

Compute $f'(\lambda_1) = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n) + 0 + 0 + \cdots + 0 \neq 0$.

Similarly, $f'(\lambda_2) \neq 0, \ldots, f'(\lambda_n) \neq 0$.

$\Rightarrow$ $f(x)$ and $f'(x)$ have no common root.

7.

Equivalently (By Problem 6),

We check if $f(x)$ and $f'(x)$ have a common root.

$$f(x) = x^4+2x^3+3x^2+2x+1$$
$$f'(x) = 4x^3+6x^2+6x+2.$$

4次 4+3=7.

3次

①. ~~the~~ Sylvester matrix

$$S_{f,f'} = \begin{pmatrix} 1 & & & 2 & & & \\ 2 & 1 & & 6 & 2 & & \\ 3 & 2 & 1 & 6 & 6 & 2 & \\ 2 & 3 & 2 & 4 & 6 & 6 & 2 \\ 1 & 2 & 3 & 0 & 4 & 6 & 6 \\ 0 & 1 & 2 & 0 & 0 & 4 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 \end{pmatrix}$$

7×7

We did not cover Sylvester matrix this semester, so it won't be on the exam. However, if you are interested in it, you may refer to the lecture note of Week 12 in 2019SMath104.

Use Sage or any method to compute ~~&~~ $\det(S_{f,f'})=0$

$\Rightarrow f$ has a multiple root.

②. 輾轉相除法.

$\frac{1}{4}x + \cancel{\frac{1}{8}}$	$x^4+2x^3+3x^2+2x+1$	$4x^3+6x^2+6x+2.$	$\frac{16}{3}x$
	$x^4+\frac{3}{2}x^3+\frac{3}{2}x^2+\frac{1}{2}x$	$4x^3+4x^2+4x$	
$\frac{1}{8}$	$\frac{1}{2}x^3+\frac{3}{2}x^2+\frac{3}{2}x+1$	$2x^2+2x+2$	$\frac{8}{3}$
$\cancel{8}$	$\frac{1}{2}x^3+\frac{3}{4}x^2+\frac{3}{4}x+\frac{1}{4}$	$2x^2+2x+2$	
	$\frac{3}{4}x^2+\frac{3}{4}x+\frac{3}{4}$	$0.$	

$\Rightarrow x^2+x+1$ is a common factor of f and f'.