Sample Questions 13

Let $\mathbf{J}_n$ be the $n \times n$ all-ones matrix. Let $\mathbf{I}_n$ be the $n \times n$ identity matrix.

1. Let n be a positive integer and $\omega = e^{\frac{2\pi}{n}i}$. Let $\mathbf{Q} = \left[\omega^{(j-1)(k-1)}\right]$. That is, the j, k-entry of $\mathbf{Q}$ is $\omega^{(j-1)(k-1)}$. Show that $\frac{1}{\sqrt{n}}\mathbf{Q}$ is a unitary matrix. [This matrix is used for the Fast Fourier Transform.]

2. Let
$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}.$$
Find S_k for $k = 0, 1, 2$ and then use them to find the characteristic polynomial of $\mathbf{A}$.

3. Let
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$
Find S_k for $k = 0, 1, 2, 3$ and then use them to find the characteristic polynomial of $\mathbf{A}$.

4. Let $\mathbf{A} = \mathbf{J}_n$. Find S_k for $k = 0, \ldots, n$ and then use them to find the characteristic polynomial of $\mathbf{J}_n$.

5. Let $\mathbf{A} = \mathbf{J}_n - \mathbf{I}_n$ be the $n \times n$ all-ones matrix. Use Problem 4 to find the characteristic polynomial of $\mathbf{A}$.

6. Let $p(x)$ be a polynomial. Let $\mathbf{A}$ be a matrix and $\mathbf{Q}$ an invertible matrix. Show that $\mathbf{Q}^{-1}p(\mathbf{A})\mathbf{Q} = p(\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q})$.

7. Show that the Cayley–Hamilton theorem is true for diagonalizable matrices. That is, if $\mathbf{A}$ is a diagonalizable matrix with its characteristic polynomial $p(x)$, then $p(\mathbf{A}) = \mathbf{O}$.