Sample Questions 12

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

1. Let $t : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto V. Find the eigenvalues and their corresponding eigenspaces. Find the characteristic polynomial of t.

2. Let $t : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection along V. Find an orthonormal basis of $\mathbb{R}^3$ and $T := \text{Rep}_{\mathcal{B},\mathcal{B}}(t)$ such that T is diagonal.

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3. Prove that similar matrices have the same characteristic polynomials.

4. Find the characteristic polynomial of $\mathbf{J}_n$, the $n \times n$ all-ones matrix.

5. Let
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$
Find an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}^\top \mathbf{A} \mathbf{Q}$ is upper triangular.

6. Prove that if $\mathbf{U}$ is an upper triangular matrix, then $\mathbf{U}\mathbf{U}^* = \mathbf{U}^*\mathbf{U}$ implies $\mathbf{U}$ is diagonal.

7. Let
$$\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$
Find the spectral decomposition
$$\mathbf{A} = \sum_{k=1}^{3} \lambda_k \mathbf{v}_k \mathbf{v}_k^\top$$
such that $\mathbf{v}_k$ is an eignevector of $\mathbf{A}$ with respect to λ_k for $k = 1, 2, 3$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthonormal.