Sample Questions 11

Let $\mathbf{I}_n$ be the $n \times n$ identity matrix. Let $\mathbf{O}_{m,n}$ be the $m \times n$ zero matrix.

1. Let V and W be two vector spaces with dimensions n and m, respectively. Let $f : V \to W$ be a homomorphism from V to W. Find a basis $\mathcal{B}$ of V and a basis $\mathcal{D}$ of W such that

$$\text{Rep}_{\mathcal{B},\mathcal{D}}(f) = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix},$$

where $r = \text{rank}(f)$.

2. Diagonalize

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

or show that it is not diagonalizable.

3. Diagonalize

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

or show that it is not diagonalizable.

4. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the characteristic polynomial of $\mathbf{A}$ and the eigenspace for each of the eigenvalues of $\mathbf{A}$.

5. Diagonalize

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

or show that it is not diagonalizable.

6. Find a 5×5 matrix whose eigenvalues are 1 and 2 such that 1 has algebraic multiplicity 2 and geometric multiplicity 1, while 2 has algebraic multiplicity 3 and geometric multiplicity 2.

7. A (left) stochastic matrix is a non-negative square matrix such that each column sum is 1. Show that every stochastic matrix has the eigenvalue 1. [In a Markov chain with the stochastic matrix $\mathbf{M}$, if $\mathbf{Mv} = \mathbf{v}$, then $\mathbf{v}$ is the final stationary distribution and is the eigenvector corresponding to the eigenvalue 1.]