國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（二） **MATH 104 / GEAI 1209: Linear Algebra II**

第二次期中考 May 4, 2020 **Midterm 2**

姓名 Name : solution

學號 Student ID # : ____________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	7 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } V = \operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_2\}).$$

(a) [2pt] Find two vectors in $V^{\perp}$ such that they are independent.

Solve $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \vec{v} = \vec{0}$

$\vec{v}$ can be $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ and they are indep.

(b) [1pt] Find a vector that is not in $V^{\perp}$.

$\vec{v}_1 \notin V^{\perp}$.

2. Let

$$A = \begin{bmatrix} i & j & k & \ell \\ m & n & o & p \\ q & r & s & t \\ u & v & w & x \end{bmatrix}.$$

By the permutation expansion, $\det(A) = insx + \cdots$ is the sum of 4! terms.

(a) [1pt] Find a term of $\det(A)$ with positive sign (other than $insx$).

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(b) [1pt] Find a term of $\det(A)$ with negative sign.

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3. Let

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 3y + 2z = 0 \right\}.$$

(a) [2pt] Find a basis of V.

y, z are free.

$y=1,\ z=0 \Rightarrow \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$

$y=0,\ z=1 \Rightarrow \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

basis $= \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(b) [3pt] Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find the projection of $\mathbf{u}$ onto the plane V.

Let $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The projection of $\vec{u}$ onto Colspace(A) ~~is~~

$$= A(A^TA)^{-1}A^T\vec{u}$$

$$= \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{7}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 6 \\ 6 & 5 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{14}\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix}$$

$$A^T\vec{u} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$(A^TA)^{-1}A^T\vec{u} = \frac{1}{14}\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

4. [5pt] Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}.$$

Apply the Gram–Schmidt algorithm to the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and obtain an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

G-S algorithm.

$$\vec{u}_1 = \vec{v}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \operatorname{proj}_{\vec{u}_1}(\vec{v}_2) = \vec{v}_2 - \frac{1}{2}\cdot\vec{u}_1$$

$$= \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\\1\\0\end{pmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \operatorname{proj}_{\vec{u}_1}(\vec{v}_3) - \operatorname{proj}_{\vec{u}_2}(\vec{v}_3$$

$$= \vec{v}_3 - 0 - \frac{1}{3/2}\cdot\vec{u}_2$$

$$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}-1/2\\1/2\\1\\0\end{pmatrix} = \begin{pmatrix}1/3\\-1/3\\1/3\\0\end{pmatrix}.$$

5. [5pt] Let

$$A = \begin{bmatrix} x & 3 & 0 & 0 \\ 3 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix},$$

where x is a real number. Find $x \in \mathbb{R}$ such that $\det(A) = 0$.

By Laplace expansion,

$$\det(A) = x \cdot \det\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} - 3 \cdot \det\begin{pmatrix} 3 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= -2x - 3 \cdot (-3)$$

$$= -2x + 9$$

If $-2x+9=0$, then $\underline{\underline{x = \frac{9}{2}}}$.

6. [5pt] Let

$$A = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}.$$

Show that $\det(A) = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$. Make sure to justify every step of your argument.

See ver. A.

7. [5pt] Let J_n be the $n \times n$ all-ones matrix. Let I_n be the $n \times n$ identity matrix. Find $\det(J_n - I_n)$ as a formula of n.

See ver. A.

8. [extra 2pt] Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Find $\det(A)$.

See ver. A .

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	2	
Total	30 (+2)	