Math589 Midterm1

6 questions, 24 total points
Note: Use other papers to answer the problems. Remember to write down your **name** and your **student ID #**.

1. [4pt] Show that every simple graph G must have two vertices whose degrees are the same.

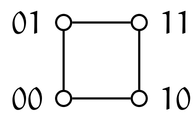
   Solution. Let G be a simple graph on n vertices. The degree of a vertex can be $0, 1, \ldots, n-1$. Suppose the degree of every vertex is different. Since there are n vertices, there is exactly one vertex of degree k for each $k = 0, 1, \ldots, n-1$. Let u be the vertex of degree 0. Then u is not adjacent to any vertex. Let v be the vertex of degree $n-1$. Then v is adjacent to any vertex. This is a contradiction since u and v can not occur simultaneously.

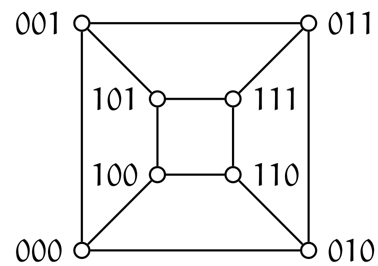
2. [4pt] Suppose G is a connected simple graph on n vertices and m edges. Show that $m \geqslant n-1$.

Solution. We prove the contrapositive statement. Start from n isolated vertices. (That is, a graph on n vertices without any edge.) When an edge is added, it at most combines two components into one component. This operation decreases the number of components by at most one. If $m \leqslant n-2$ edges were added, then the number of components is at least $n-m \geqslant 2$, so G is not connected.

3. [4pt] Find all connected graphs on 5 vertices. How many of them?
   Solution. There are 21 connected graphs on 5 vertices.

4. [4pt] The *Hamming distance* between two 0, 1-strings is the number of different digits. For example, the Hamming distance between 010101 and 111000 is 3. The *hypercube* H_d of dimension d has vertices as all 0, 1-strings of length d, and two vertices are adjacent if the Hamming distance of the strings is 1. The graphs below illustrate H_2 and H_3. Find a partition $X \dot{\cup} Y = V(H_d)$ so that every edge of H_d is in between X and Y.



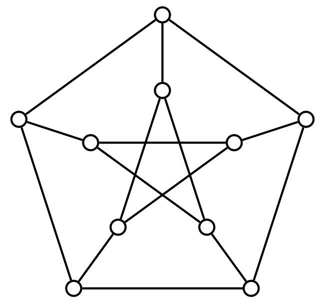


Solution. Partition $V(H_d)$ into two parts V_{odd} and V_{even}, where V_{odd} is all the 0, 1-strings in $V(H_d)$ with odd number of ones and V_{even} is all the strings with even number of ones. Thus, all edges are between these two sets.

5. [4pt] Let C_n be the cycle graph on n vertices and L_n the Laplacian matrix of C_n. Recall that $L_n(1,1)$ is the matrix obtained from L_n by removing the first row and the first column. Compute $|\det L_n(1,1)|$.

   Solution. The cycle graph C_n has n spanning trees, so $|\det L_n(1,1)| = n$ by the matrix-tree theorem.

6. Let G be the graph below. Find the chromatic number $k = \chi(G)$ and give a proper k-coloring of G.



Solution. It contains an odd cycle, so $\chi(G) \geqslant 3$. And it has a 3-coloring as below. Therefore, $\chi(G) = 3$.