Math589 Homework 9

Note: To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Let F be a subset of $\{e_1, -e_1, \dots, e_d, -e_d\}$. Show that $\text{conv}(F)$ is a proper face of the crosspolytope if and only if there is no i such that both e_i and $-e_i$ are in F.

   Solution. Suppose $e_i, -e_i \in F$. Let $h : v \cdot x = b$ be a hyperplane with $v, x \in \mathbb{R}^d$ and $b \in \mathbb{R}$. Let $v = (v_1, \dots, v_d)$. If h passes through the two points e_i and $-e_i$, then $v_i = -v_i = b$. Therefore, $v_i = 0 = b$. Without loss of the generality, we may assume that $v_1 \neq 0$. Thus, e_1 and $-e_1$ are on the different sides of h, so $\text{conv}(F)$ is not a proper face of the crosspolytope.

   Suppose there is no i such that $e_i, -e_i \in F$. Let v be the sum of all vectors in F. Then $v \cdot e = 1$ for all $e \in F$. For all $e \in \{e_1, -e_1, \dots, e_d, -e_d\}$ other than those in F, $v \cdot e = 0$. So $\text{conv}(F)$ is a proper face of the crosspolytope.

2. Let

$$v_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \text{ and } v_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Let Δ be the simplicial complex composed of the simplices

$$\operatorname{conv}(\{v_0, v_1, v_2\}), \operatorname{conv}(\{v_0, v_2, v_3\}), \operatorname{conv}(\{v_0, v_3, v_4\}), \operatorname{conv}(\{v_0, v_4, v_1\}),$$

and their faces. Define $f : V(\Delta) \to \mathbb{R}^2$ by

$$f(v_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, f(v_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, f(v_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, f(v_3) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \text{ and } f(v_4) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a exact formula for the affine extension $\|f\|$ of f. That is, what is

$$\|f\|(\begin{bmatrix} x \\ y \end{bmatrix})?$$

Solution. This one is easy. Graphically, $\|f\|$ is the 45°-rotation along with the $\sqrt{2}$-scaling. Therefore,

$$\|f\|(\begin{bmatrix} x \\ y \end{bmatrix}) = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$