Sample Questions 5

1. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$. Write down the definition of the *inner product* $\mathbf{u} \cdot \mathbf{v}$. Go through each definition in Chapter 1 of the textbook. Think about how you are going to write the answer if it is asked in the exam.

2. Write down a 3×3 singular matrix and a 3×3 nonsingular matrix. For each properties that we have learned, think about an example with the property and an example without the property.

3. Let
$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \\ 1 \\ 1 \end{bmatrix}.$$
Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

4. Prove that a matrix
$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
is nonsingular if and only if the determinant $ad - bc$ is nonzero.

5. Let $\mathbf{A}$ be a 3×3 matrix. Prove that $\mathbf{A}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$. (This is also true for any $n \times n$ matrix, but we need to learn the meaning of the determinant first.)

6. Let $\mathbf{A}$ be a 3×3 matrix. Your goal is to give $k \leqslant 9$ entries of $\mathbf{A}$ so that $\mathbf{A}$ is nonsingular no matter what the remaining entries are. What is the minimum k to achieve this? (This is the 3×3 version of the game we played in the class.)

7. Let
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 & 1 \\ 1 & 2 & -4 & 2 \\ 2 & 3 & -6 & 5 \\ 3 & 3 & -9 & 4 \end{bmatrix}$$
and let $\mathbf{e}_j \in \mathbb{R}^4$ be the vector whose j-th entry is 1 while other entries are zero. For each $j = 1, 2, 3, 4$, solve $\mathbf{A}\mathbf{v}_j = \mathbf{e}_j$ for $\mathbf{v}_j$. Is there a better way than doing the row operations four times?