

Sample Solutions for Sample Questions 12

1. (a) $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is linear.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

~~$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 + y_1 \\ x_1 + y_1 \\ x_1 + y_1 \end{pmatrix}$$~~

$$f\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$$

$$f\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + f\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ x_2 \\ x_2 \end{pmatrix} \xrightarrow{\quad} \text{same.}$$

$$f\left(r \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) = f\left(\begin{pmatrix} rx_1 \\ ry_1 \\ rz_1 \end{pmatrix}\right) = \begin{pmatrix} rx_1 \\ rx_1 \\ rx_1 \end{pmatrix}$$

$$r \cdot f\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) = r \begin{pmatrix} x_1 \\ x_1 \\ x_1 \end{pmatrix} \xrightarrow{\quad} \text{same.}$$

(b) Yes, an isomorphism is a homomorphism.

(c) No, $f: \mathbb{R} \rightarrow \mathbb{R}$ is not a homomorphism.
 $x \mapsto x^3$

$$\left. \begin{array}{l} f(1+1) = f(2) = 8 \\ f(1)+f(1) = 1+1 = 2 \end{array} \right\} \text{not the same.}$$

2.

$$\left[\begin{array}{l} \text{Key: Write } \begin{bmatrix} 9 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} , \\ \text{Then } f\left(\begin{bmatrix} 9 \\ 8 \end{bmatrix}\right) = c_1 (1+x+x^2) + c_2 (-2+3x+x^2) \end{array} \right]$$

$$\text{Solve } \begin{bmatrix} 9 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} .$$

$$\Rightarrow \begin{cases} c_1 + 3c_2 = 9 \\ 2c_1 + c_2 = 8 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = 2 \end{cases}$$

$$\text{So } f\left(\begin{bmatrix} 9 \\ 8 \end{bmatrix}\right) = f\left(3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$$

$$= 3 \cdot f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + 2 \cdot f\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$$

$$= 3(1+x+x^2) + 2(-2+3x+x^2)$$

$$= -1 + 9x + 5x^2 .$$

3.

Assumption: $\begin{cases} B = \{\vec{v}_1, \dots, \vec{v}_k\} \text{ is linearly indep.} \\ f: V \rightarrow W \text{ is a homomorphism.} \\ f \text{ is one-to-one} \end{cases}$

[Method: Suppose $c_1 f(\vec{v}_1) + \dots + c_k f(\vec{v}_k) = \vec{0}$.
Show $c_1 = \dots = c_k = 0$.]

Suppose $c_1 f(\vec{v}_1) + \dots + c_k f(\vec{v}_k) = \vec{0}$

Since f is a homomorphism,

$$f(c_1 \vec{v}_1 + \dots + c_k \vec{v}_k) = \vec{0}.$$

Since f is one-to-one and $f(\vec{0}) = \vec{0}$,

it must be $c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$.

Since B is indep.,

$$c_1 = \dots = c_k = 0.$$

Goal: $f(B)$ is indep.

4.

Assumption: $\begin{cases} \text{span}(B) = V, & B = \{\vec{v}_1, \dots, \vec{v}_k\}. \\ f: V \rightarrow W \text{ is a homomorphism} \\ f \text{ is onto.} \end{cases}$

[Method: For each $\vec{w} \in W$, find c_1, \dots, c_k]
such that $c_1 f(\vec{v}_1) + \dots + c_k f(\vec{v}_k) = \vec{w}$.]
Suppose $\vec{w} \in W$

Since f is onto,

there exist $\vec{v} \in V$ such that $f(\vec{v}) = \vec{w}$.

Since $\text{span}(B) = V$, there are c_1, \dots, c_k

such that $c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{v}$.

Thus,

$$f(c_1 \vec{v}_1 + \dots + c_k \vec{v}_k) = f(\vec{v}) = \vec{w}.$$

Since f is a homomorphism,

$$\vec{w} = f(c_1 \vec{v}_1 + \dots + c_k \vec{v}_k) = c_1 f(\vec{v}_1) + \dots + c_k f(\vec{v}_k)$$

Goal: $\text{span}(f(B)) = W$.

5.

$$T: \mathcal{P}_3 \longrightarrow \mathcal{P}_3$$

$$a + bx + cx^2 + dx^3 \longmapsto b + 2cx + 3dx^2$$

$$\text{range}(f) = \{ b + 2cx + 3dx^2 \mid a, b, c, d \in \mathbb{R} \}$$

= 全部的二次(含以下)多项式.

$$= \mathcal{P}_2.$$

$$\text{rank} = \dim \text{range}(f) = \dim \mathcal{P}_2 = 3.$$

$$\text{nullspace}(f) = \{ a + bx + cx^2 + dx^3 \mid \begin{array}{l} b + 2cx + 3dx^2 = 0 \\ a, b, c, d \in \mathbb{R} \end{array} \}$$

$$= \{ a \mid a \in \mathbb{R} \}.$$

$$= \mathcal{P}_0.$$

$$\text{nullity} = \dim \text{nullspace}(f) = \dim \mathcal{P}_0 = 1.$$

For 6, 7,

see the proof in the lecture notes.

Pretending you are taking the exam,

practice how to write it down.