Sample Questions 12

For the following problems, $\mathcal{P}_d$ is the space of all polynomials with real coefficients and of degree at most d.

1. For each function f in Problem 4 of SampleQuestion11, check if f is linear. That is, check if f is a homomorphism or not.

2. Suppose $f : \mathbb{R}^2 \to \mathcal{P}_2$ is a homomorphism with $f\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = 1 + x + x^2$ and $f\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = -2+3x+x^2$. Find $f\left(\begin{bmatrix}9\\8\end{bmatrix}\right)$.

3. Suppose $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set in V and $f : V \to W$ is a homomorphism that is one-to-one. Show that $f(B)$ is a linearly independent set in W.

4. Suppose $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a spanning set in V and $f : V \to W$ is a homomorphism that is onto. Show that $f(B)$ is a spanning set in W. That is, if $\text{span}(B) = V$ then $\text{span}(f(B)) = W$.

5. Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be the derivative operation with $T(x^k) = kx^{k-1}$. Find the range space $\text{range}(T)$ and the null space $\text{nullspace}(T)$. Compute the rank and the nullity of T.

The following two problems are in the lecture notes. You may write it again in your own words and make sure you understand the logic of every steps.

6. Let $f : V \to W$ be a homomorphism. Show that $f(X)$ is a subspace of W if X is a subspace of V.

7. Let $f : V \to W$ be a homomorphism. Show that $f^{-1}(Y)$ is a subspace of V if Y is a subspace of W.