國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

第一次期中考 October 14, 2019 **Midterm 1**

姓名 Name : solution

學號 Student ID # : ____________________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page, **6 pages** of questions, score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**25 points** + 2 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Write down an example of a system of **linear** equations in variables a, b, and c.

$$\begin{cases} a+b+c=0 \\ a+b+c=1 \end{cases}$$

2. [1pt] Write down an example of a system of equations in variables a, b, and c that is **not a linear system**.

$$\begin{cases} a^2+b^2+c^2=0 \\ c=3. \end{cases}$$

3. [1pt] Write down an example of a system of **three linear equations** in its **echelon form** that contains **two free variables**.

$$\begin{cases} a \qquad +d+e=0 \\ \quad b \quad +d+e=0 \\ \qquad c+d+e=0. \end{cases}$$

4. [1pt] Write down an example of a 4×4 **nonsingular** matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. [1pt] Write down an example of a 4×4 **singular** matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

6. Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ \sqrt{5} \\ 0 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a) [1pt] Find the length $|\mathbf{u}|$.

$$|\vec{u}| = \sqrt{2^2 + 3^2 + \sqrt{5}^2 + 0^2}$$
$$= \sqrt{4+9+5} = \sqrt{18}$$

(b) [1pt] Find the length $|\mathbf{v}|$.

$$|\vec{v}| = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}.$$

(c) [1pt] Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\vec{u} \cdot \vec{v} = 3 + 0 = 3.$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|} = \frac{3}{\sqrt{18}\sqrt{2}} = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

(d) [2pt] Find a vector $\mathbf{w}$ such that the angle between $\mathbf{w}$ and $\mathbf{v}$ is $\frac{\pi}{4}$. [The answer is not unique. You only need to find one.]

e.g. $\vec{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

7. [5pt] Find the general solution of the following linear system.

$$\begin{cases} w - 2x - 2y + 7z = -12 \\ -2w + 4x + 5y - 19z = 28 \\ -4w + 8x + 11y - 43z = 60 \end{cases}$$

That is, find $\mathbf{p}$ and $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_k$ such that

$$\{\mathbf{p} + c_1\boldsymbol{\beta}_1 + \cdots + c_k\boldsymbol{\beta}_k : c_1, \dots, c_k \in \mathbb{R}\}$$

is the set of all solutions.

$$\left(\begin{array}{cccc|c} 1 & -2 & -2 & 7 & -12 \\ -2 & 4 & 5 & -19 & 28 \\ -4 & 8 & 11 & -43 & 60 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & -2 & -2 & 7 & -12 \\ 0 & 0 & 1 & -5 & 4 \\ 0 & 0 & 3 & -15 & 12 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & -2 & -2 & 7 & -12 \\ 0 & 0 & 1 & -5 & 4 \end{array}\right) \rightsquigarrow \underbrace{\begin{array}{c} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -4 \\ & & 1 & -5 & 4 \end{array}\right) \\ \begin{array}{cccc} & \uparrow & & \uparrow \end{array} \\ \text{free} \end{array}}_{R\vec{v} = \vec{r}}$$

- Solve $R\vec{v} = \vec{r}$ for $\vec{p}$.

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -3 & -4 \\ & & 1 & -5 & 4 \end{array}\right) \\ \begin{array}{cccc} & \uparrow & & \uparrow \\ & 0 & & 0 \end{array} \end{array} \Rightarrow \underline{\underline{\vec{p} = \begin{pmatrix} -4 \\ 0 \\ 4 \\ 0 \end{pmatrix}}}$$

- Solve $R\vec{v} = \vec{0}$ for $\vec{\beta}_1, \vec{\beta}_2$.

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -3 & 0 \\ & & 1 & -5 & 0 \end{array}\right) \\ \begin{array}{cccc} & \uparrow & & \uparrow \\ & 1 & & 0 \end{array} \end{array} \Rightarrow \underline{\underline{\vec{\beta}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}}$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & -2 & 0 & -3 & 0 \\ & & 1 & -5 & 0 \end{array}\right) \\ \begin{array}{cccc} & \uparrow & & \uparrow \\ & 0 & & 1 \end{array} \end{array} \Rightarrow \underline{\underline{\vec{\beta}_2 = \begin{pmatrix} 3 \\ 0 \\ 5 \\ 1 \end{pmatrix}}}$$

8. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & -18 \\ -1 & -1 & -1 & 3 \\ 4 & 4 & 5 & -7 \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

It is known that $\mathbf{R}$ is the reduced echelon form of $\mathbf{A}$. Write the row $\begin{bmatrix} 1 & 0 & 0 & -3 \end{bmatrix}$ as a linear combination of rows of $\mathbf{A}$.

$$\begin{pmatrix} 1 & 2 & -1 & -18 \\ -1 & -1 & -1 & 3 \\ 4 & 4 & 5 & -7 \end{pmatrix} \xrightarrow[-4\rho_1+\rho_3]{\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & -1 & -18 \\ 0 & 1 & -2 & -15 \\ 0 & -4 & 9 & 65 \end{pmatrix} \xrightarrow{4\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & -1 & -18 \\ 0 & 1 & -2 & -15 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

$$\xrightarrow[\rho_3+\rho_1]{2\rho_3+\rho_2} \begin{pmatrix} 1 & 2 & 0 & -13 \\ & 1 & 0 & -5 \\ & & 1 & 5 \end{pmatrix} \xrightarrow{-2\rho_2+\rho_1} \begin{pmatrix} 1 & & & -3 \\ & 1 & & -5 \\ & & 1 & 5 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 1 \\ & 1 & 2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ -4 & & 1 \end{pmatrix} A = R.$$

$$\begin{pmatrix} 1 & -2 & -3 \\ & 1 & 2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 4 & 1 \end{pmatrix} A = R.$$

$$\begin{pmatrix} -1 & -14 & -3 \\ 1 & 9 & 2 \\ 0 & 4 & 1 \end{pmatrix} A = R.$$

$$(-1)\cdot(1 \;\; 2 \;\; -1 \;\; -18) + (-14)\cdot(-1 \;\; -1 \;\; -1 \;\; 3)$$
$$+ (-3)(4 \;\; 4 \;\; 5 \;\; -7) = (1 \;\; 0 \;\; 0 \;\; -3)$$

9. [5pt] Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 9 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{bmatrix}.$$

Is it possible to obtain **B** from **A** by some row operations? [This is a yes-or-no question, but you have to provide your ~~answer~~ reason.]

Yes, since both A and B have the same reduced echelon form.

10. [extra 2pt] It is known that the following row operations are correct.

$$\begin{bmatrix}13\\23\end{bmatrix} \xrightarrow{-\rho_1+\rho_2} \begin{bmatrix}13\\10\end{bmatrix} \xrightarrow{-\rho_2+\rho_1} \begin{bmatrix}3\\10\end{bmatrix} \xrightarrow{-3\rho_1+\rho_2} \begin{bmatrix}3\\1\end{bmatrix}$$

Find two integers a and b such that $a \cdot 13 + b \cdot 23 = 1$.

See ver. A .

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	2	
Total	25 (+2)	