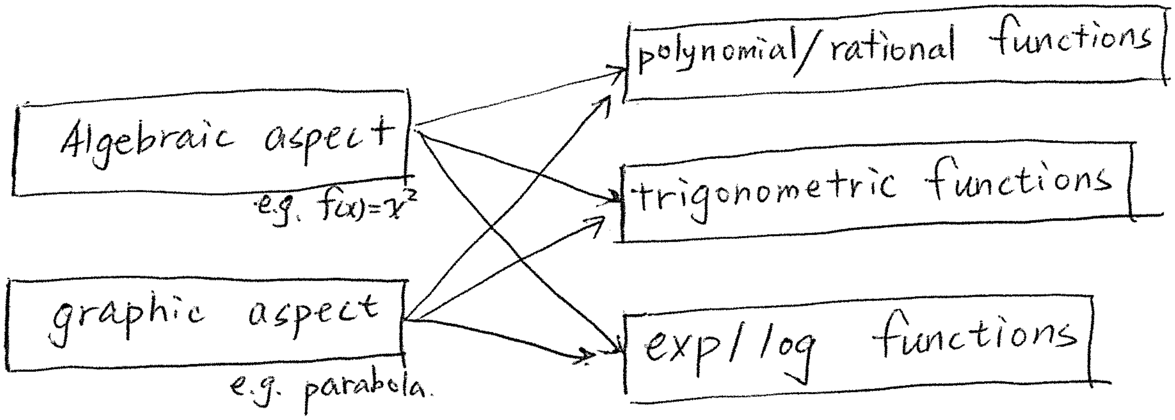
Math 120 : Precalculus.

- Course outline on Course Spaces
  - grading scheme
  - HW, quizzes, midterms, final.
  - HW in MML.
- Check out MyMathLab.

Objective: Learn the different aspects of <u>functions</u>,
and how to manipulate them.



§ 1.1 Real numbers.

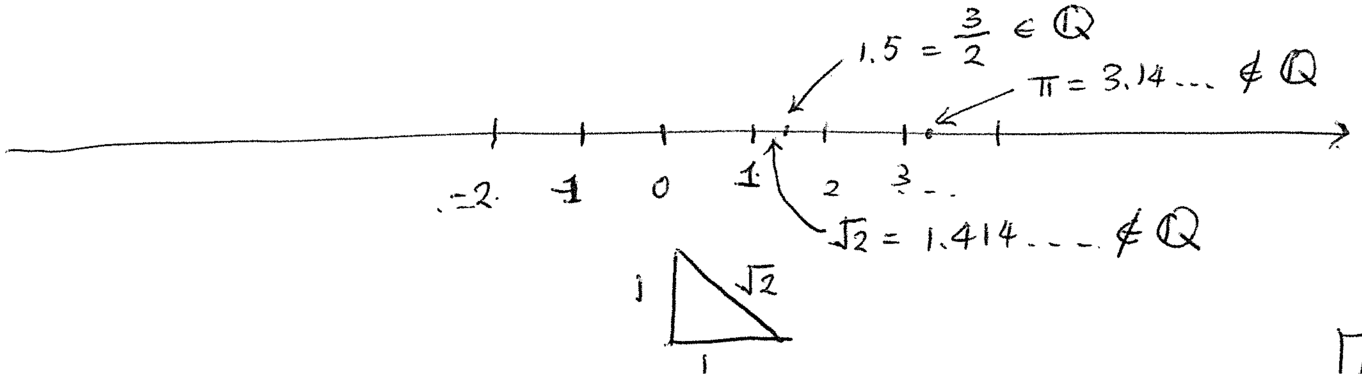
Sets:

counting numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. (good with $+$)

integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (good with $+, -$)

rational numbers $\mathbb{Q} = \{\frac{a}{b} : a, b \text{ are integers}\}$ (good with $\times, \div$).

real numbers $\mathbb{R} = \{\text{every point on the line}\}$ (filling all gaps).



Operations : PEMDAS

first ↓

parentheses () : always go first.] meta level

exponential a^b : $a \times a \times \cdots \times a$ for b times] level 3

multiplication $\overset{b\times a}{\cancel{a\times b}}$: $a + a + \cdots + a$ for b times.] level 2

division $b \div a$: $b \times \frac{1}{a}$ (the reverse work of ×)] level 2

addition $b+a$:] level 1.

substraction $b-a$: $b+(-a)$ (the reverse work of +)] level 1.

last.

- in the same level: commutative and associative.

$$a-b+c = a+(-b)+c = a+c+(-b) = a+c-b.$$

$$a-b+c = (a-b)+c = a+(-b+c) \neq a-(b+c)$$

e.g
$$\begin{array}{ccc} 3-2+1 & \neq & 3-(2+1) \\ \parallel & & \parallel \\ 1+1 & & 3-3 \\ \parallel & & \parallel \\ 2 & & 0 \end{array}$$

- in different levels: distributive.

$$a\times(b+c) = a\times b + a\times c$$

e.g.
$$\begin{array}{ccc} 3\times(2+1) & = & 3\times 2 + 3\times 1 \\ \parallel & & \parallel \\ 3\times 3 & & 6+3 \\ \parallel & & \parallel \\ 9 & & 9 \end{array}$$

- Notations for multiplication : $3\times x = 3\cdot x = 3x$, means 3 copies of x.
  $3\times 2 = 3\cdot 2 \neq 32$

- minus means "reversed" plus

$b-a = b+(-a)$

$-a = -1\cdot a$

$-(-a) = a$

$\Rightarrow -(a-b) = -a-(-b) = -a+b$

relations & orders :

- ~~Between~~ For every two real numbers a and b, either $a>b$, $a=b$, or $a<b$. [trichotomy property].
- In short we als write $a \geq b$ or $a \leq b$.
- equal sign $=$ has the properties:
  - $a=a$ [reflexive]
  - If $a=b$, then $b=a$ [symmetric]
  - If $a=b$ and $b=c$, then $a=c$ [transitive].
  - $a=$ ~~m~~ b means you can substitute a by b in any case.

  e.g. I "am" human.

  e.g. lecture note $=$ textbook (X).

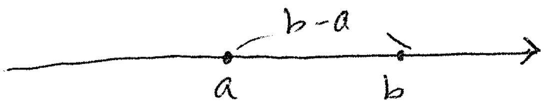
absolute value :

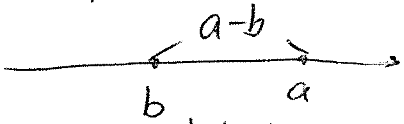
$$|a| = \begin{cases} a & \text{if } a \geq 0; \\ -a & \text{if } a<0 \end{cases}.$$

e.g. $|3|=3$, $|-1|=1$, $|-500|=500$

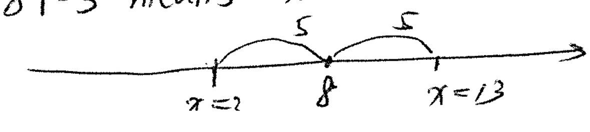
※ abs value is the "signless" value of the number.

- properties :
  - $|a| \geq 0$
  - $|-a|=|a|$
  - $|a \cdot b| = |a| \cdot |b|$ and $\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$
  - $|a-b|$ means the distance between points a and b.

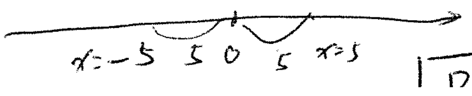




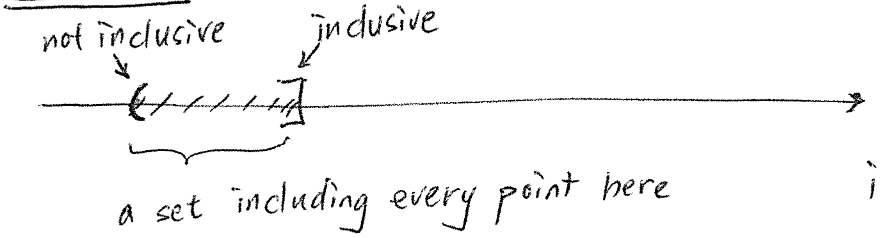
eg. $|x-8|=5$ means $x=3$ or $x=13$.



Q: $|x|=5$ means $|x-0|=5$



intervals :



$[-3, 5]$ closed interval

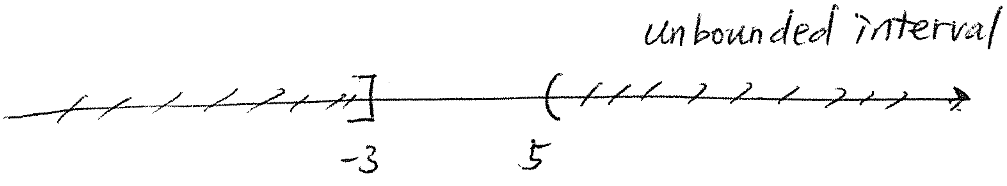
$(-3, 5)$ open interval

$(-3, 5]$ interval

e.g. $5 \overset{\text{inside}}{\in} [-3, 5]$

$5 \notin (-3, 5)$

$5 \in (-3, 5]$



$(-\infty, -3]$ $(5, \infty)$

inequalities :

$$x > 3 \xrightarrow{\text{solution}} x \in (3, \infty)$$

$$x \leq 5 \longrightarrow x \in (-\infty, 5]$$

- solve inequality :

  [same as solving equality $(+, -, \overbrace{\times}^{\text{nonzero}}, \div$ same amount on both sides$)$

  [but $\times$ or $\div$ a negative number <u>should</u> reverse the inequality .

e.g. Solve $-3x - 9 > 0$.

$-3x > 9$ $[+9]$

$x < -3$ $[\div(-3)]$ reverse !!

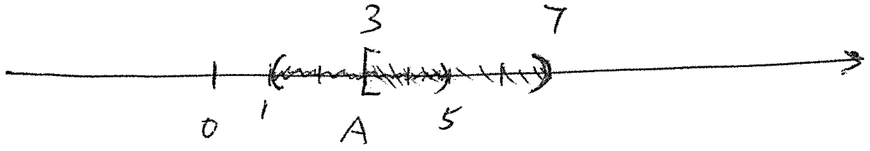
equivalently

$x \in (-\infty, -3)$.

Try : Solve $-5x + 10 \leq 0$.

(✓) $x \geq 2$ (×) $x \leq 2$.

and, or :



$A = (1,5)$, $B = [3,7)$

intersection $A \cap B = [3,5)$ means "in A" **and** "in B"

union $A \cup B = (1,7)$ means "in A" or "in B".

union is bigger.

e.g. Solve $-7 < 3x+2 < 7$

① $-7 < 3x+2$ **and** ② $3x+2 < 7$

①

$-7 < 3x+2$ [−2]

$-9 < 3x$ [÷3]

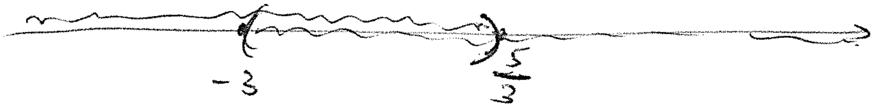
$-3 < x$

② $3x+2 < 7$

$3x < 5$ [−2]

$x < \frac{5}{3}$ [÷3]

so $x \in (-3, \infty)$ **and** $x \in (-\infty, \frac{5}{3})$.



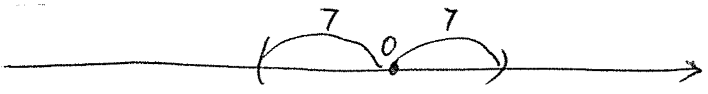
$\Rightarrow x \in (-3, 5)$.

Try: Solve $5x-2 \geq 3$ or $5x-2 \leq -3$.

(×) no solution (✓)③ $x \geq 1$ or $x \leq -\frac{1}{5}$

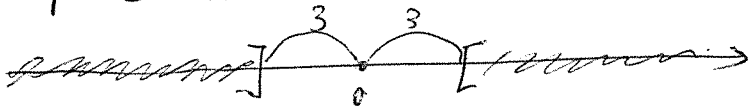
abs inequalities : write abs into compound inequalities.

$|3x+2| < 7$ means $-7 < 3x+2 < 7$.



within distance 7.

$|5x-2| \geq 3$ means $5x-2 \leq -3$ or $5x-2 \geq 3$.



distance ≥3

Try:
Solve $|x-3| \leq 2$.
(✓) [1,5] (×)(0,6)

§1.3 Equations and graphs.

variable: a changeable value; e.g., age, balance, time

- often has restrictions, or relations.

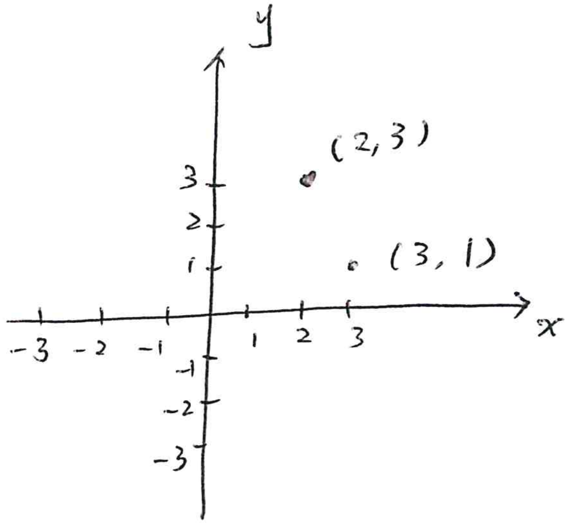
  e.g., age ≥ 0, age = year − 1987

  age ← year : age depends on year

  balance ← day.

coordinate system of two variables

- usually x and y
- each point is a pair

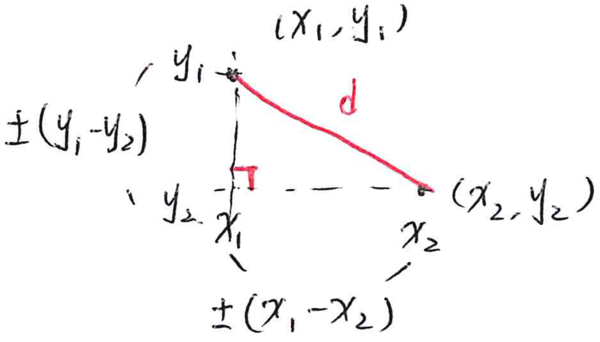


- between two points (x_1, y_1), (x_2, y_2)
  - the distance is

    $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

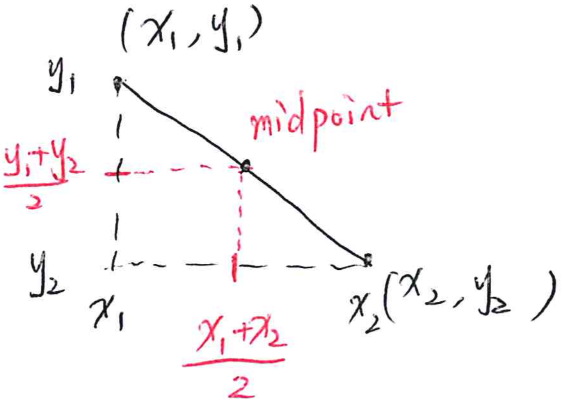
  - the midpoint is

    $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$



Pythagor~~as~~ean theorm

$d^2 = (y_1-y_2)^2 + (x_1-x_2)^2$



graph of an equation

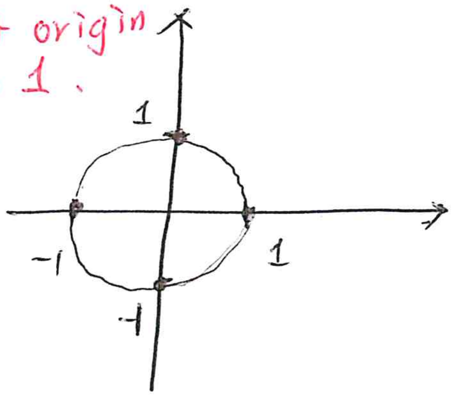
- means draw all points that satisfies the equation.

e.g., $x^2+y^2=1 \Leftrightarrow$ a circle centred at origin with radius 1.

points include $(\pm 1, 0)$, $(0, \pm 1)$

and all points (x,y) with

$x^2+y^2=1$ (magnitude = 1)

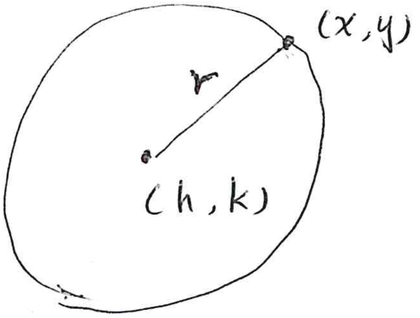


circle:

- standard form

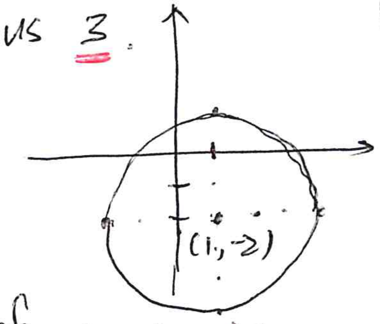
$$(x-h)^2+(y-k)^2=r^2$$

$\Leftrightarrow$ a circle centred at (h,k) with radius r.



e.g. Sketch $(x-1)^2+(y+2)^2=\underline{9}$.

$\Rightarrow$ a circle centred at $(1,-2)$ with radius $\underline{3}$.



e.g. Find the equation of a circle centred at $(-3,5)$ and passing through $(1,8)$.

centre : $(-3,5)$

radius : $\sqrt{(-3-1)^2+(5-8)^2} = 5$

so

$$(x+3)^2+(y-5)^2=5^2$$

- find the standard form by completing the square

e.g. Sketch $x^2+6x+y^2-5y=-\frac{1}{4}$

note: $x^2+6x+9=(x+3)^2$

$y^2-5y+\frac{25}{4}=(y-\frac{5}{2})^2$

So $x^2+6x+\underline{\underline{9}}+y^2-5y+\underline{\underline{\frac{25}{4}}}=-\frac{1}{4}+9+\frac{25}{4}=15$

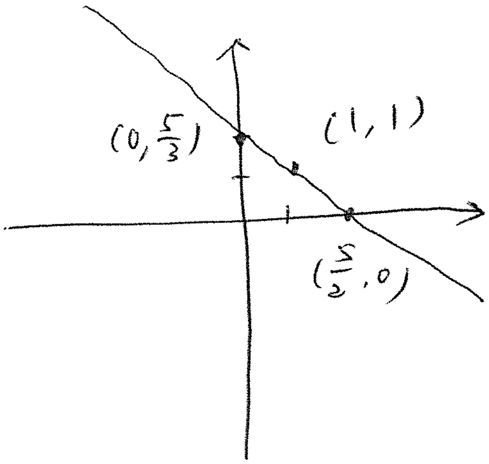
$\Rightarrow (x+3)^2+(y-\frac{5}{2})^2=15$

$\Rightarrow$ a circle centred at $(-3,\frac{5}{2})$ with radius $\underline{\sqrt{15}}$

Completing the square:

$x^2+bx+(\frac{b}{2})^2=(x+\frac{b}{2})^2$

$x^2-bx+(\frac{b}{2})^2=(x-\frac{b}{2})^2$



Lines:

- Standard form

$$Ax+By=C.$$

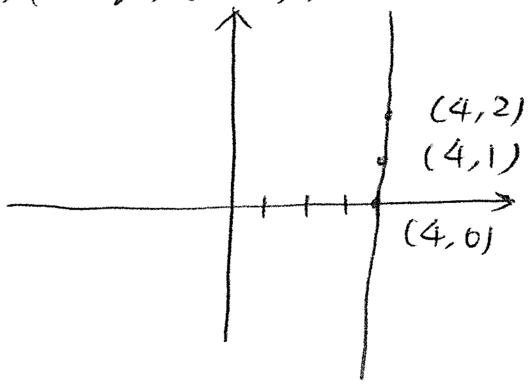
e.g., $2x+3y=5$

points include $(1,1)$, $(0,\frac{5}{3})$, $(\frac{5}{2},0)$

e.g., $x=4$

points include $(4,0)$, $(4,1)$, $(4,2)$, ...

$\Rightarrow$ vertical line



- intercepts
  - x-intercept: the point $(x_0, 0)$ where the line and x-axis intersect
  - y-intercept : $(0, y_0)$
    ~ y-axis

e.g. $2x+3y=5$.

If $x=0 \Rightarrow 3y=5 \Rightarrow y=\frac{5}{3} \Rightarrow$ y-intercept $(0, \frac{5}{3})$

If $y=0 \Rightarrow 2x=5 \Rightarrow x=\frac{5}{2} \Rightarrow$ x-intercept $(\frac{5}{2}, 0)$.

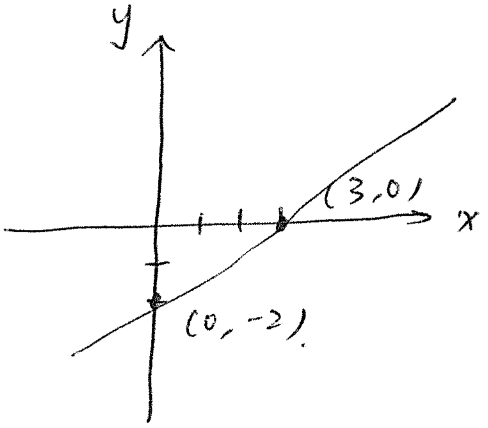
- two - intercept form

$$\frac{x}{a}+\frac{y}{b}=1$$

has x-intercept $(a, 0)$
y-intercept $(0, b)$

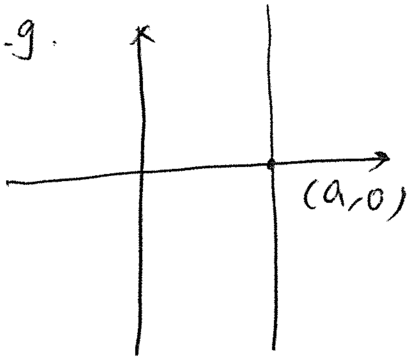
e.g. Find the equation of

x-intercept $(3,0)$
y-intercept $(0,-2)$



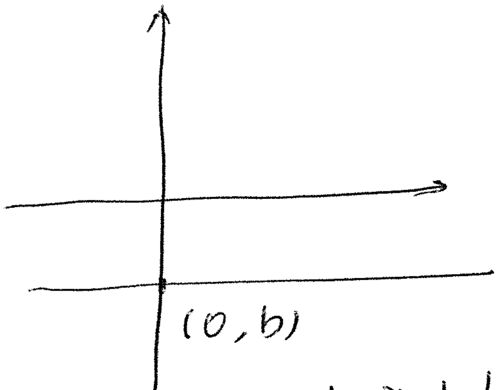
$\Rightarrow$ equation $\frac{x}{3}+\frac{y}{-2}=1 \Leftrightarrow 2x-3y=6$ [×6]

e.g.



no y-intercept (vertical)

$\frac{x}{a}=1$ or $x=a$



no x-intercept (horizotal)

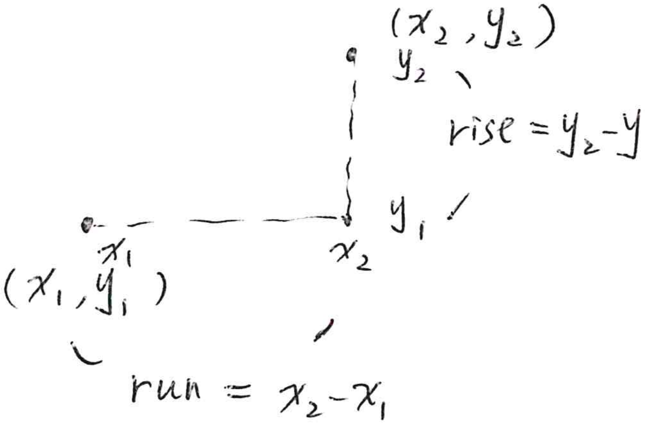
$\frac{y}{b}=1$ or $y=b$.

§ 1.4 Lines

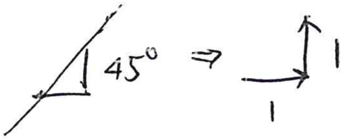
slope:

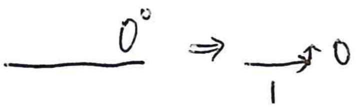
- the slope of the line through (x_1, y_1) and (x_2, y_2)

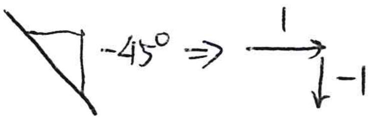
is $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$



- slope determines the angle

▪  ⇒ slope = 1

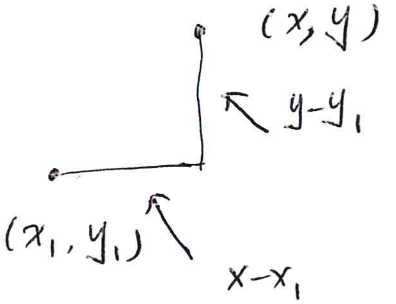
▪  ⇒ slope = 0

▪  ⇒ slope = −1.

Point-slope form:

- the line through (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1)$$



so $m = \frac{y - y_1}{x - x_1}$

Slope-intercept form:

- the line ~~through~~ with slope m and y-intercept $(0, b)$

is $$y = mx + b.$$

[Use point-slope form
$y - b = m(x - 0) \Leftrightarrow y = mx + b$.]

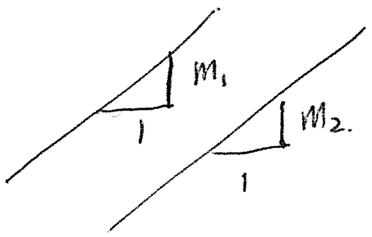
Summary:

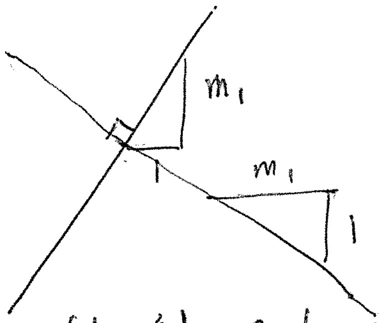
a line can be determined by

- two intercepts (points) → two-intercept form
- slope and a point (intercept) → point-slope form (slope-intercept form)

slope and angle:

- two lines with slopes m_1 and m_2 is
  - parallel if $m_1 = m_2$
  - perpendicular if $m_1 \cdot m_2 = -1$





$\Rightarrow m_2 = \frac{-1}{m_1}$

e.g., Find the line through $(1,-4)$ and parallel to $y=3x+2$.

parallel, so slope = slope of $y=3x+2 = 3$.

point = $(1,-4)$

$\Rightarrow$ point-slope form $(y+4) = 3 \cdot (x-1)$.

e.g., Find the line through $(1,-4)$ and perpendicular to $y=3x+$

perpendicular, so slope $= -\frac{1}{3}$

point $= (1,-4)$

$\Rightarrow$ point-slope form $y+4 = -\frac{1}{3} \cdot (x-1)$

§ 1.5 Functions.

functions

- y is a function of x means
  y depends on x, or x determines y.

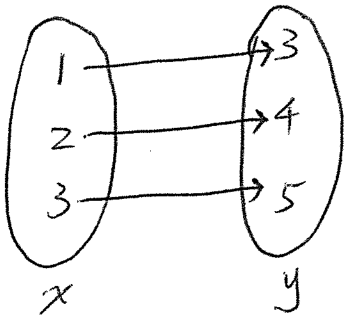
e.g., The temperature is a function of time.
The scores is a function of effort.
The abs value $|x|$ is a function of x.

- function is a set of pairs $\{(x,y)\}$ such that
  each x appears in only one pair.

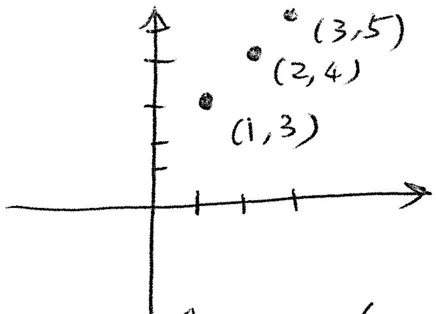
e.g., $f = \{(1,3), (2,4), (3,5)\}$

domain $= \{1,2,3\}$.

range $= \{3,4,5\}$.



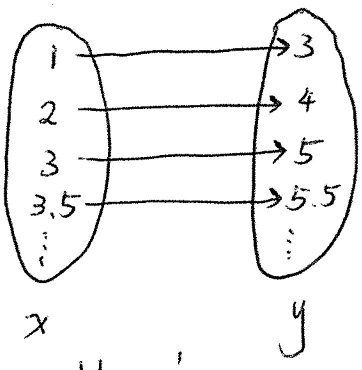
graph of f



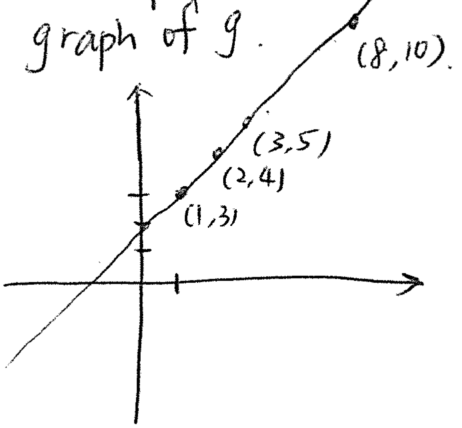
$g = \{(x, x+2) : x \in \mathbb{R}\}$

domain $= \mathbb{R}$

range $= \mathbb{R}$

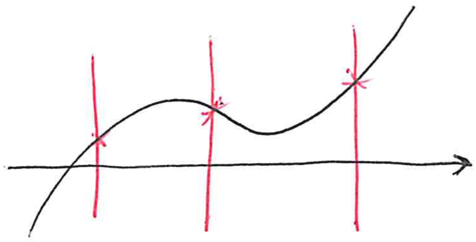


graph of g.

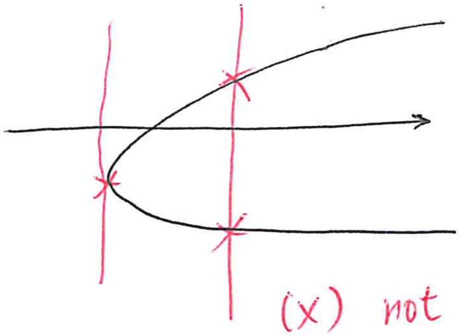


- **domain** = all possible x's
- **range** = all possible y's

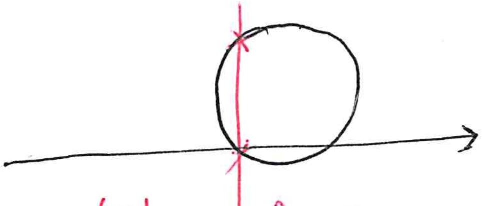
- the graph of a function draws all the pairs
  (such that each x appears in only one pair)
- vertical test :
  each vertical line touch the graph only once.



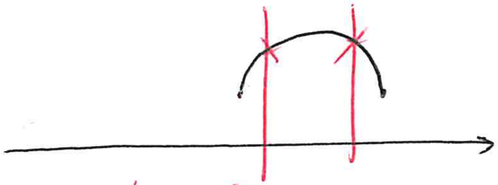
(√) function



(x) not function

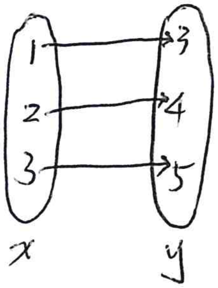


(x) not function



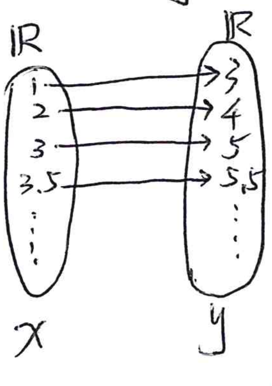
(√) function.

notation : f is a function ⇒ f describes a relation from x to y.



$\Rightarrow f(1)=3,\ f(2)=4,\ f(3)=5$

[f of one equals three]



$\Rightarrow f(x) = x+2.$ or $y = x+2$

just a placeholder

$f(\square) = \square + 2$

so $f(x^2) = x^2 + 2$

A real example:

When you throw a ball,

the height h of the ball is a function of the time t

e.g. height depends on t

say $h(t) = -t(t-20)$.

$t=0 \Rightarrow h(0) = -0\cdot(0-20) = 0$.

$t=10 \Rightarrow h(10) = -10\cdot(10-20) = 100$.

① When does the ball fall on ground (again)?

solve $h(t)=0$ [on ground $\Leftrightarrow$ height $=0$]

$-t(t-20)=0 \Rightarrow t=0$ or 20

$t=0$ ↑ ~~Start~~ start

20 ↑ fall on ground.

Ans: $t=20$.

② What is the rate of change of the height from $t=0$ to $t=10$?

$$\text{rate of change} = \frac{\text{difference of height}}{\text{difference of time}} = \frac{h(10)-h(0)}{10-0} = \frac{100}{10} = 10.$$

※: f is a function. The rate of change (of f) from a to b

is $$\frac{\text{difference of } f}{\text{difference of } x} = \frac{f(b)-f(a)}{b-a}.$$

Also called the difference ~~quo~~quotient.

many examples:

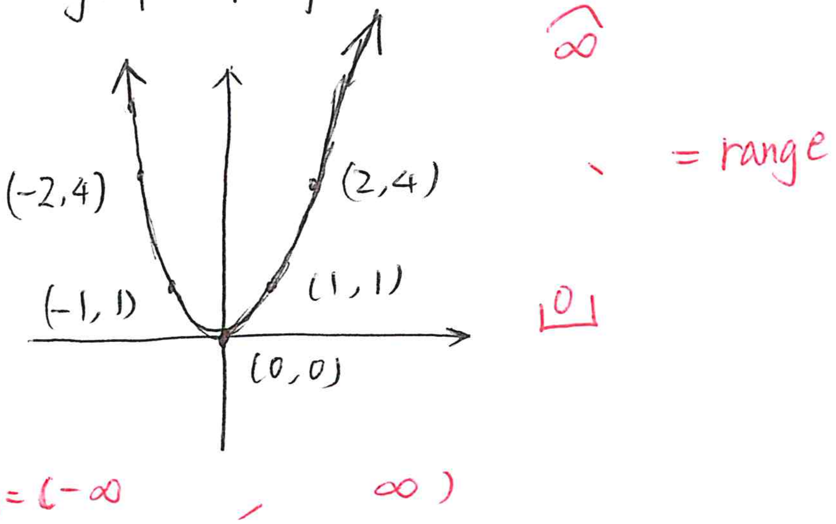
$f(x) = x^2$

x	0	1	2	-1	-2
$f(x)$	0	1	4	1	4

domain = $\mathbb{R}$

range = $[0, \infty)$

graph of f



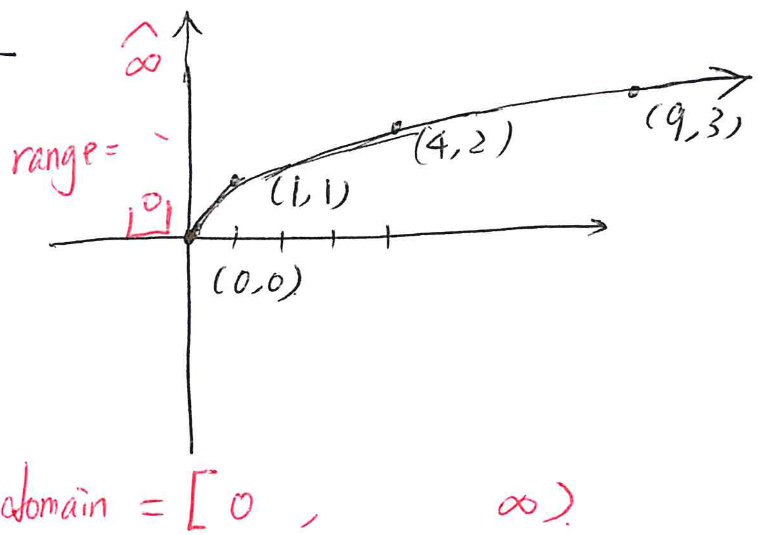
$f(x) = \sqrt{x}$

x	-1	0	1	2	3	4	⋯	9
$f(x)$	X	0	1	$\sqrt{2}$	$\sqrt{3}$	2		3

domain = $[0, \infty)$

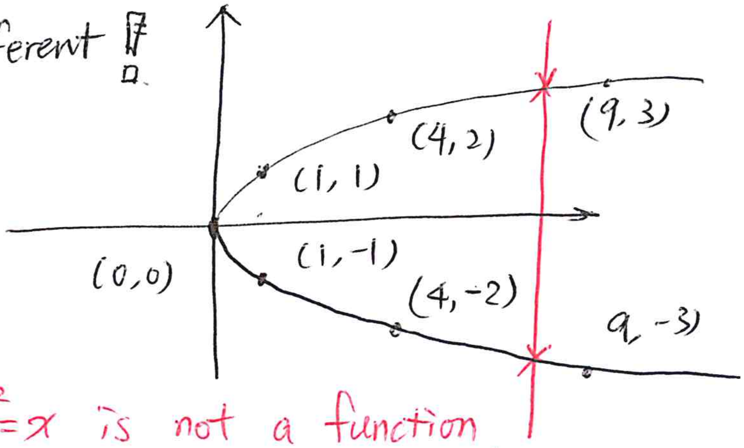
range = $[0, \infty)$.

graph of f



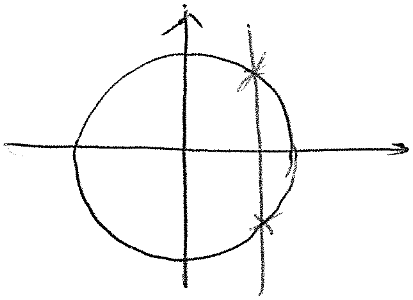
$y^2 = x$ and $y = \sqrt{x}$ are different !!

x	0	1	4	9	....
y	0	1, -1	±2	±3	



$y^2 = x$ is not a function.

$x^2+y^2=1$ not function

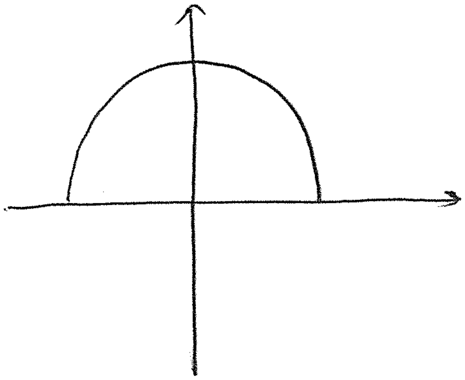


$\Downarrow$

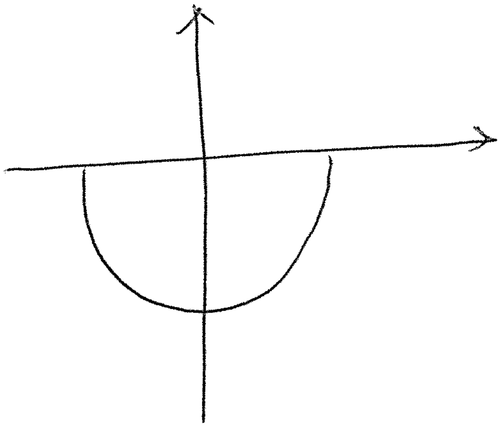
$y^2 = 1-x^2$

$\Rightarrow y = \pm\sqrt{1-x^2}$

$y=\sqrt{1-x^2}$ is function



$y=-\sqrt{1-x^2}$ is function



piecewise function

e.g.,
$$f(x) = \begin{cases} 3 & \text{if } x=1. \\ 4 & \text{if } x=2. \\ 5 & \text{if } x=3. \end{cases}$$

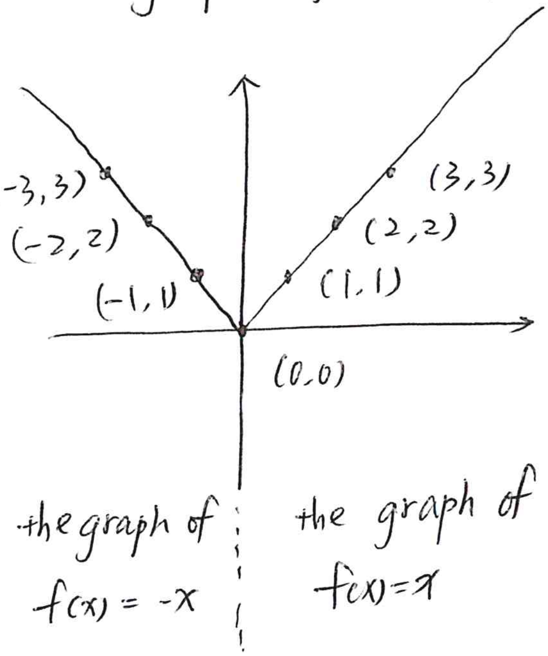
e.g.,
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

graph of $|x|$.

x	-3	-2	-1	0	1	2	3
$\lvert x\rvert$	3	2	1	0	1	2	3

increasing on $(0, \infty)$

decreasing on $(-\infty, 0)$.



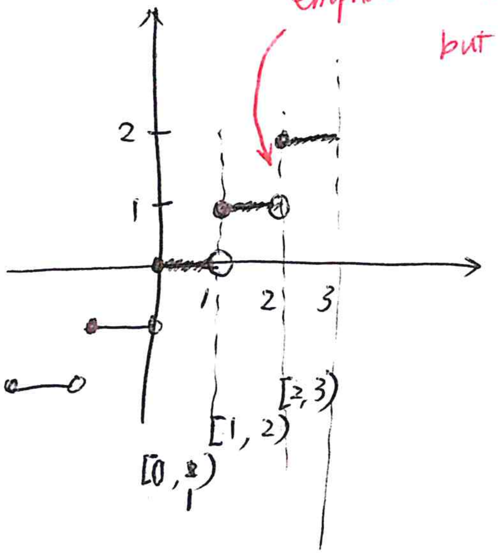
e.g., $f(x) = [x]$ ← the greatest integer $\leq x$.

so $f(1) = 1$, $f(1.5) = 1$, $f(2.1) = 2$.

$f(-2) = -2$, $f(-2.5) = -3$, $f(-2.1) = -3$.

$$f(x) = \begin{cases} 1 & \text{if } 1 \leq x < 2. \\ 2 & \text{if } 2 \leq x < 3 \\ 3 & \text{if } 3 \leq x < 4 \\ \vdots \end{cases}$$

emphasize $f(2) = 2$
but not 1

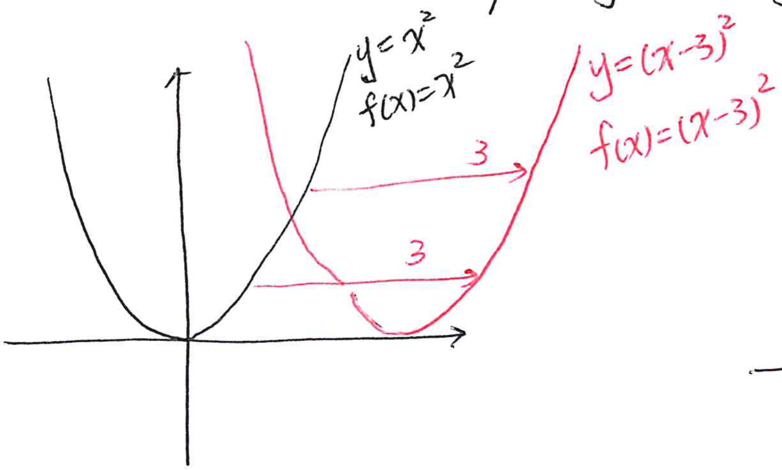


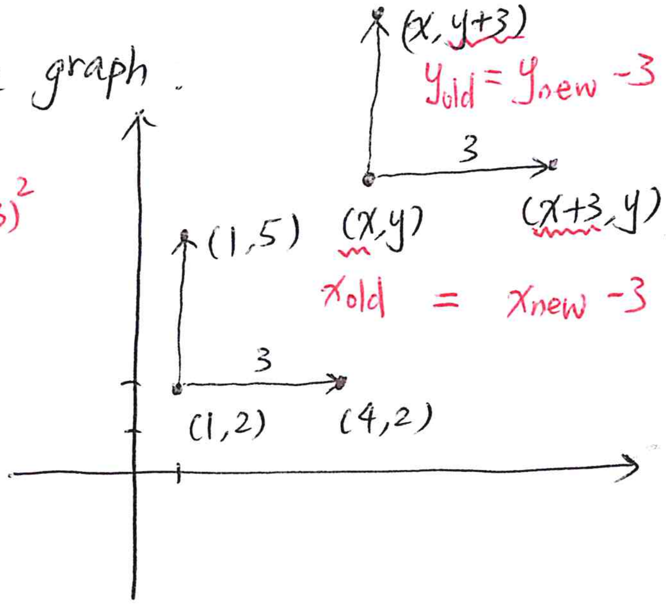
Also called stair function.

constant on $(0,1)$ or
$(1, 2)$ or
$(2, 3)$ or
$\vdots$

§1.7 Transformation.

Goal: Learn how to more/change the graph.



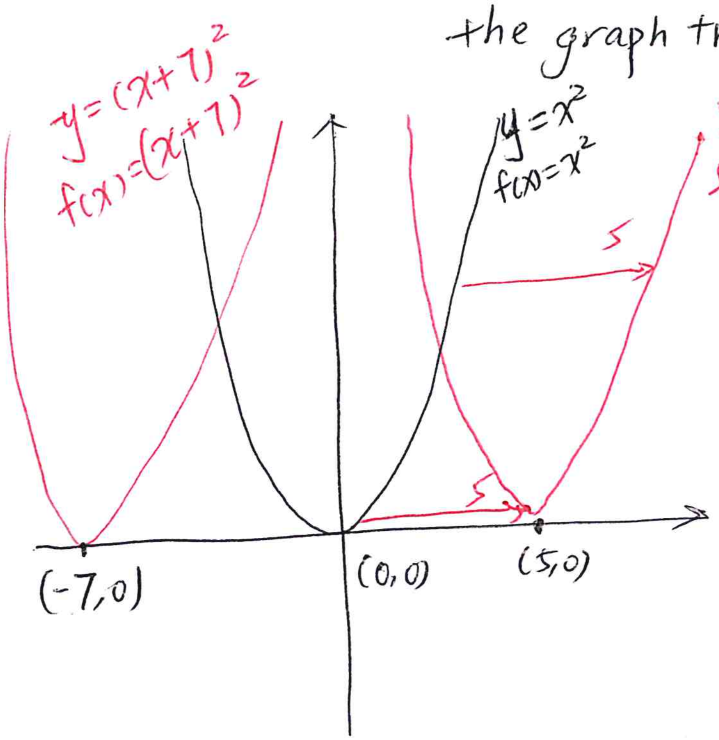


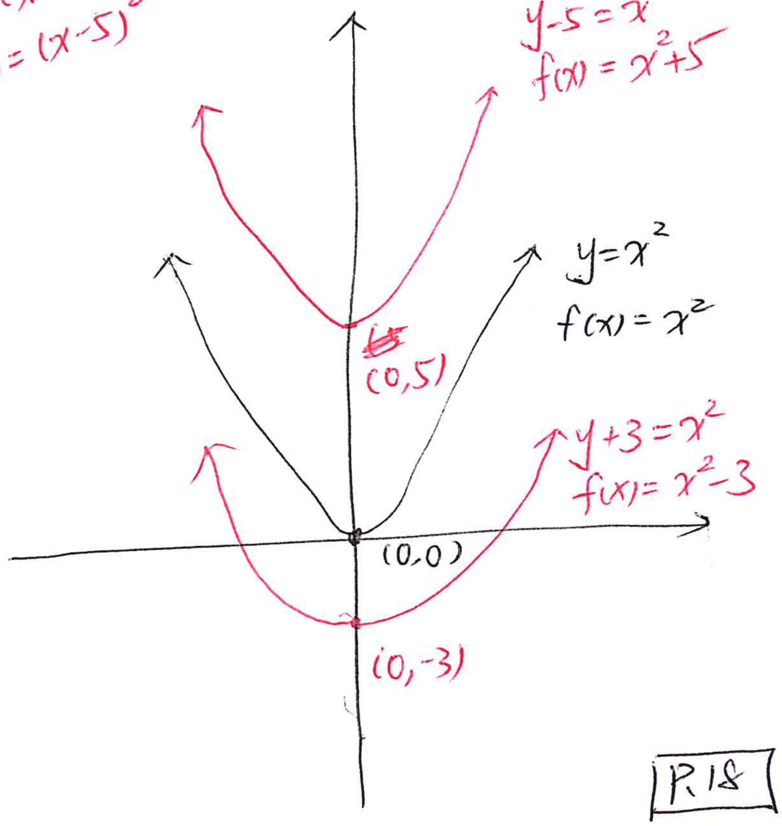
Rule of translation:

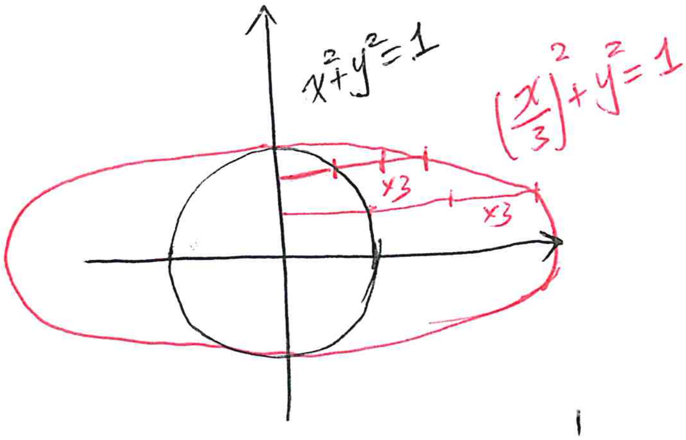
- replacing x_{old} by $x_{new}-3$ means

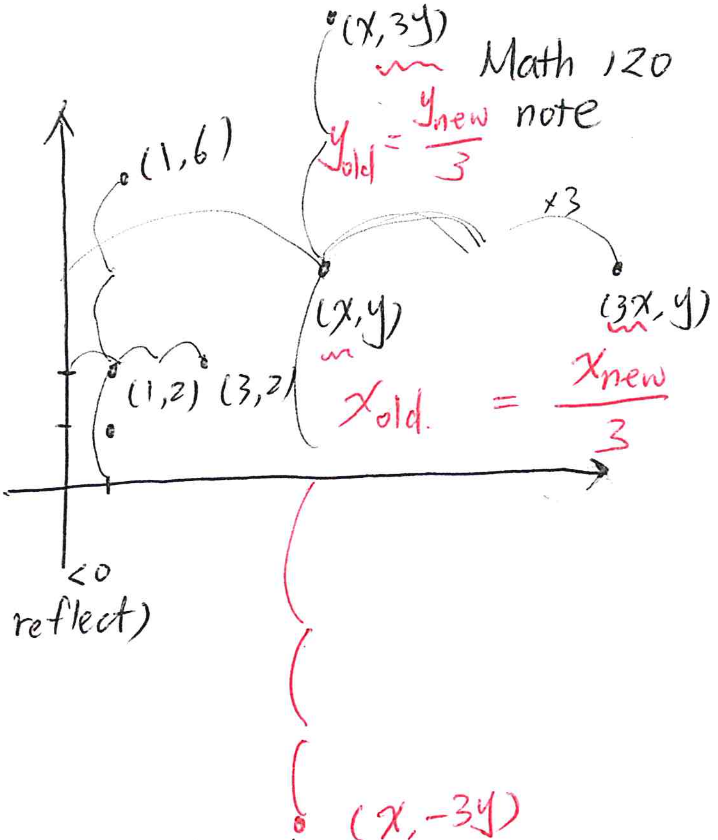
  the graph translate to the right by 3.
- replacing y_{old} by $y_{new}-3$ means

  the graph translate upward by 3.









>1 <1 <0

Rule of scaling: (strech, shrink, reflect)

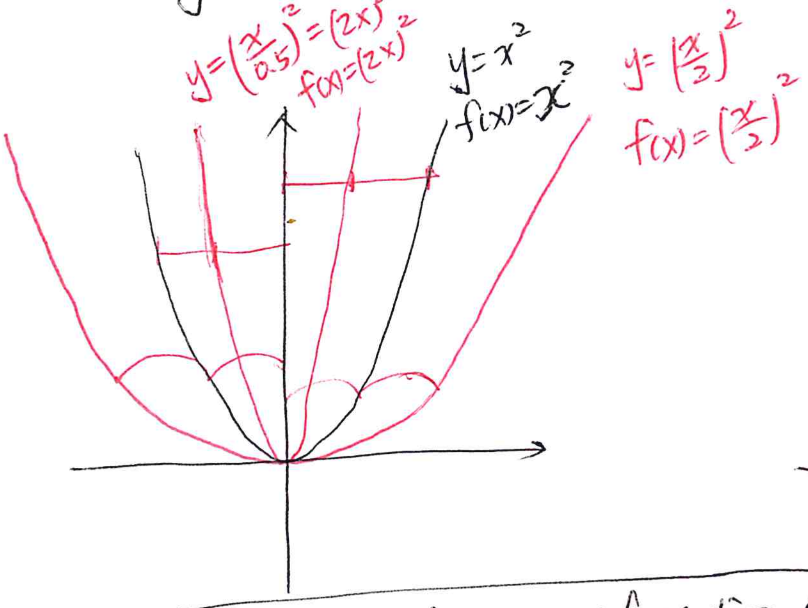
- replace x_{old} by $\frac{x_{new}}{3}$ means

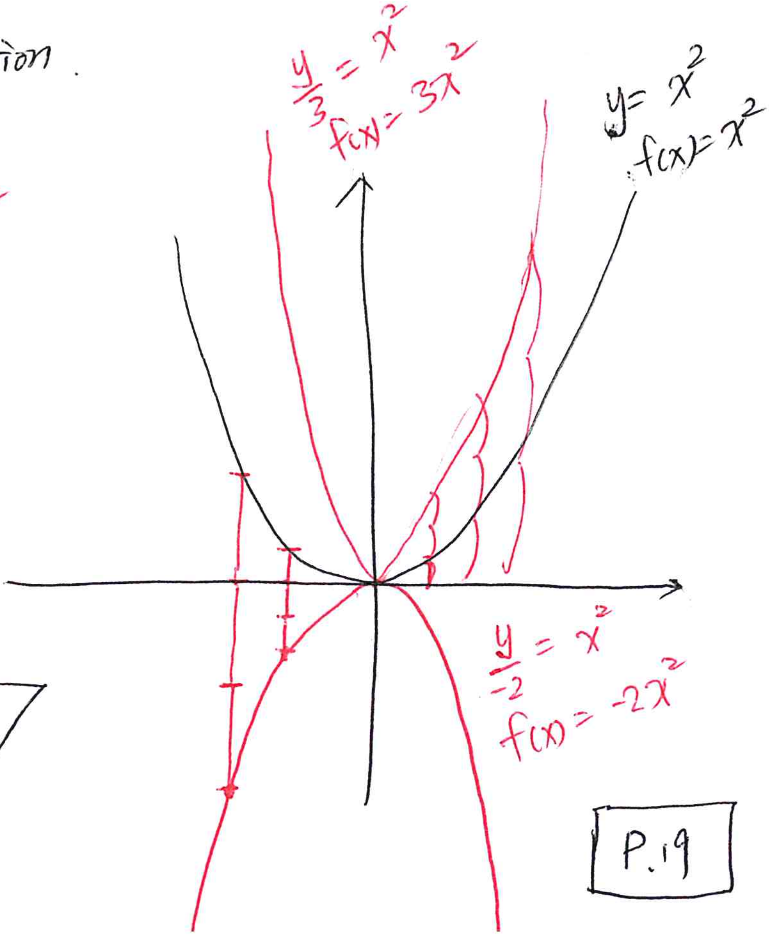
  the graph scale horizontally by 3.

- replace y_{old} by $\frac{y_{new}}{3}$ means

  the graph scale vertically by 3.

- negative number means reflection.





All parapala is a transformation of $f(x)=x^2$

Symmetry of a function:

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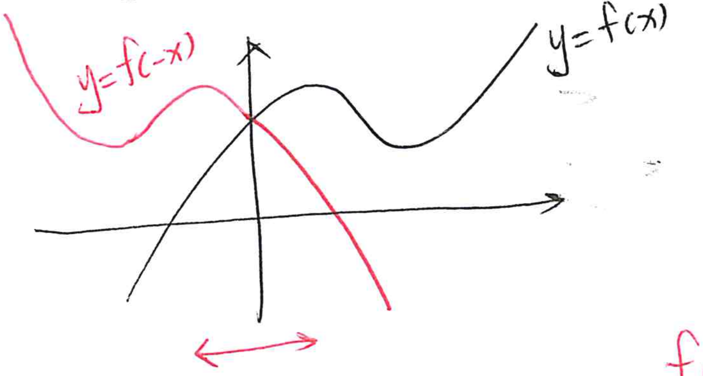
note

- symmetric by y-axis $\Longleftrightarrow$ $f(x) = f(-x)$

called "even function"



e.g. $f(x) = x^2$, $f(x) = x^4$, $f(x) = |x|$,

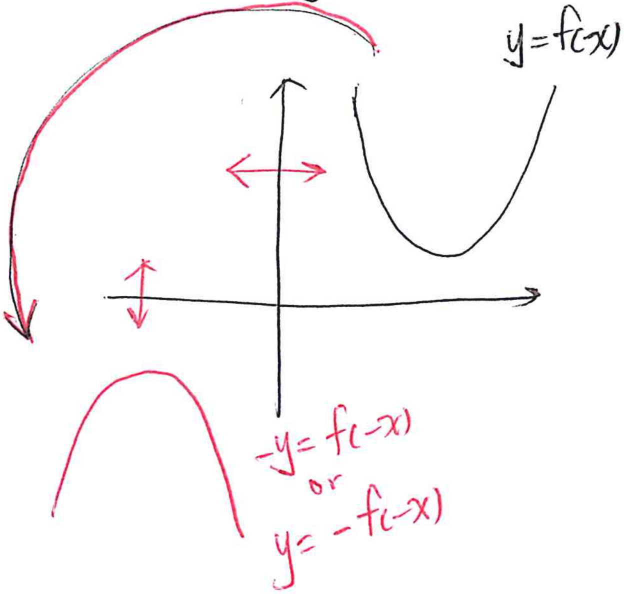
or $f(x) = x^4 - x^2$

$f(\square) = \square^4 - \square^2$

$f(-x) = (-x)^4 - (-x)^2$

$= x^4 - x^2 = f(x)$

- rotate by 180 degree (symmetric along the origin)



$\Leftrightarrow$ $f(x) = -f(-x)$

or $f(-x) = -f(x)$

called "old function"

will learn later

e.g. $f(x) = x$, $f(x) = x^3$, $f(x) = \tan x$,

or $f(x) = x^3 + x$

$f(\square) = \square^3 + \square$

~~$f(-x) = -[(-x)^3 + (-x)]$~~

$f(-x) = (-x)^3 + (-x)$

$= -x^3 - x = -(x^3 + x) = -f(x)$

Solve inequality by graph:

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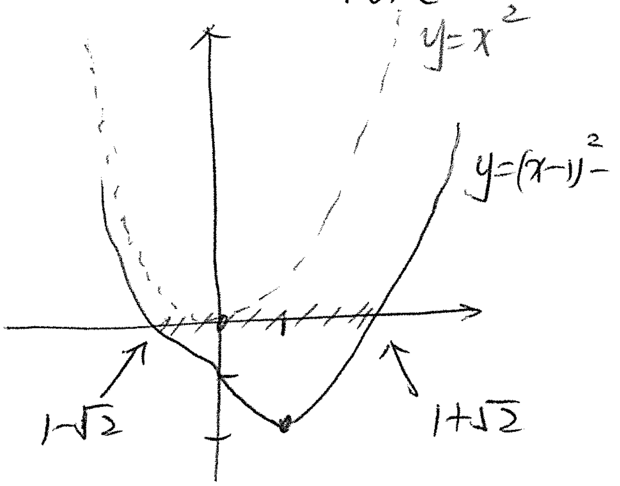
note

e.g. Solve $(x-1)^2-2<0$



- Draw $y=(x-1)^2-2$
  which come from $y=x^2$
  by $\xrightarrow{1}$ $\downarrow^{2}$
- $y<0$ means it's a point below x-axis.
- find intersection:

  if $y=0 \Rightarrow (x-1)^2-2=0$

  $(x-1)^2=2$ $[+2]$

  $x-1=\pm\sqrt{2}$ $[\sqrt{\ }]$

  $x=1\pm\sqrt{2}$ $[+1]$
- Ans: $x\in(1-\sqrt{2},\ 1+\sqrt{2})$.

§1.8 Function operations

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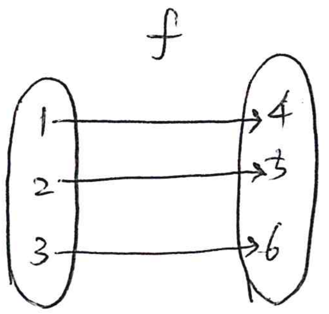
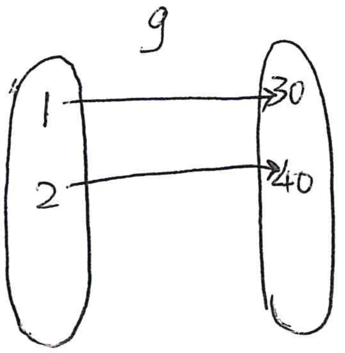
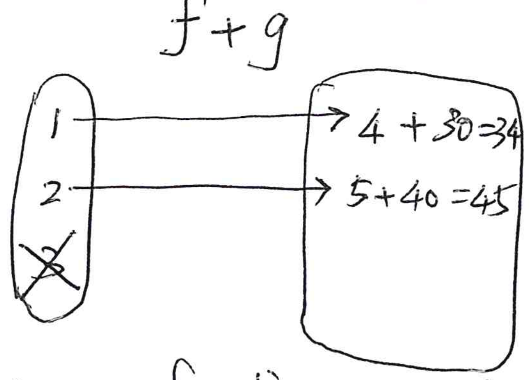
note







- $f+g$ is a new function such that

$$\underbrace{(f+g)(x)} = f(x) + g(x)$$

new function plug in x = f plug in x + g plug in x.

e.g. $f(x) = x+3$, $g(x) = x^2-1$

Then $f+g$ is a new function with

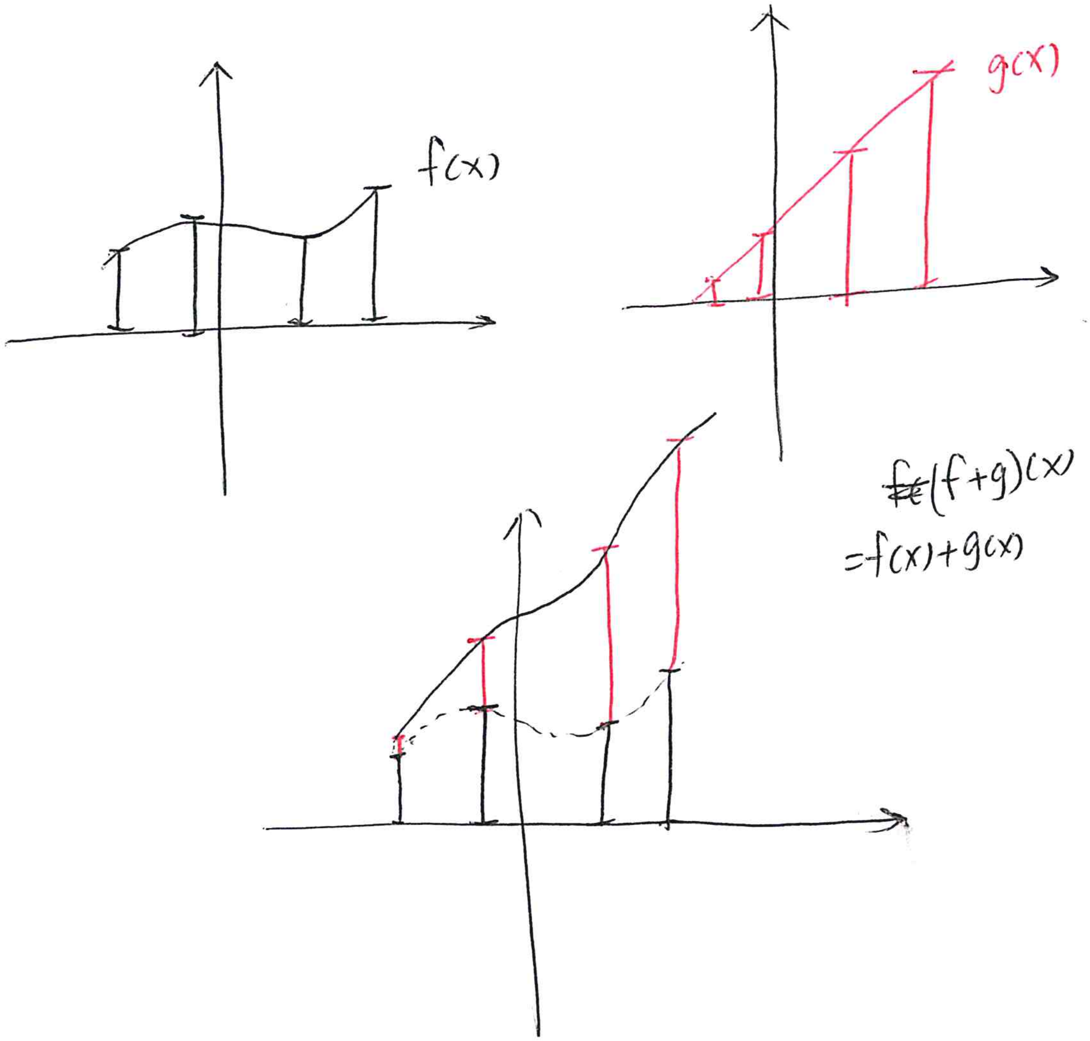
$$(f+g)(x) = x+3+x^2-1 = x^2+x+2.$$

- $+, -, \times, \div$ defined in similar way.

- for $+, -, \times$: domain of $\overset{-\text{ or }\times}{f+g}$ = (domain of f) $\cap$ (domain of g).

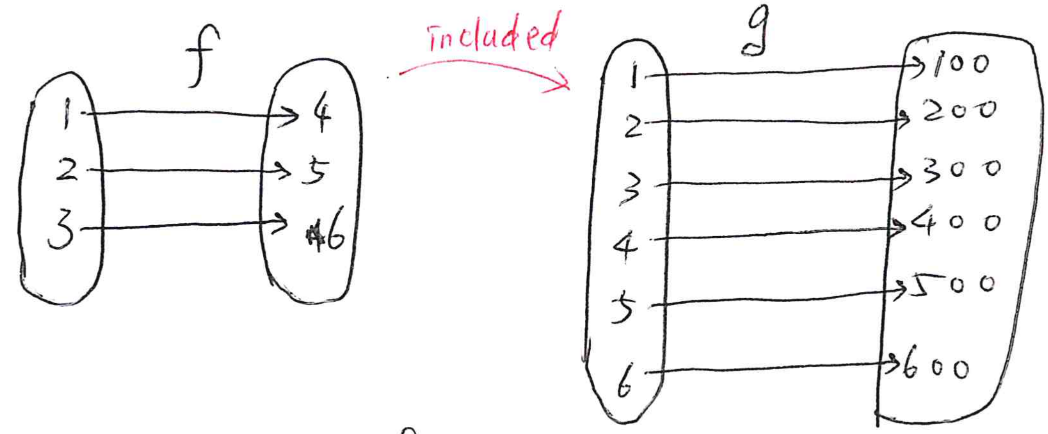
- for $\div$: domain of $\frac{f}{g}$ = (domain of f) $\cap$ (domain of g) $\cap$ $\{x : g(x) \neq 0\}$

Graphic meaning

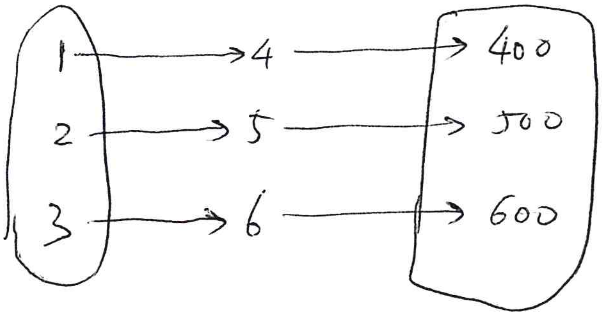


~~fe~~(f+g)(x)
=f(x)+g(x)

Composition of two functions :



$g \circ f$.



$$x \longrightarrow f(x) \longrightarrow g(f(x)) = g \circ f(x)$$

- The composition of two functions f and g is

$$g \circ f(x) = g(f(x))$$

new function
plug in x = plug in x to f ⇒ get a value $f(x)$
then plug in $f(x)$ to g.

e.g. time $\xrightarrow{\text{determine}}$ height $\xrightarrow{\text{determine}}$ temperature.
So temperature is a function of time.

e.g. $f(x) = x^2+1$, $g(x) = x-1$

- Find $f \circ g$

$f(\square) = \square^2 + 1$; $f \circ g(x) = f(g(x)) = \boxed{g(x)}^2 + 1 = (x-1)^2 + 1$
$= x^2 - 2x + 2$.

- Find $g \circ f$

$g(\square) = \square - 1$; $g \circ f(x) = g(f(x)) = \boxed{f(x)} - 1 = x^2+1 - 1$
$= x^2$

- $f \circ g$ and $g \circ f$ are in general different.

- domain of $g \circ f$ = domain of f

  domain of $f \circ g$ = domain of g.

- Understand $\underbrace{f}_{\text{framework}}(\underbrace{g(x)}_{\text{pattern}})$

e.g. $f(x) = 5x^2 + 2x + 1$

$g(x) = x^2 + 1$

$$f(g(x)) = 5\boxed{g(x)}^2 + 2\boxed{g(x)} + 1$$

$$= 5(x^2+1)^2 + 2(x^2+1) + 1$$

e.g. $h(x) = (\underline{x+1})^2 + (\underline{x+1})$; if $g(x) = x+1$,
find f such that $f(g(x)) = h(x)$.

$h(x) = \square^2 + \square$ with $\square = x+1$

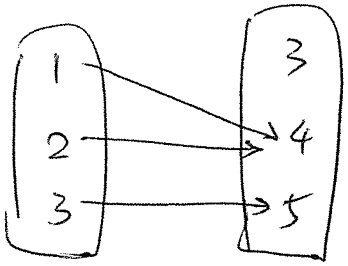
so $f(x) = x^2 + x$

e.g. $h(x) = \sqrt{x^2-1} - (x^2-1)^2$; if $f(x) = \sqrt{x} - x^2$
find g such that $f(g(x)) = h(x)$.

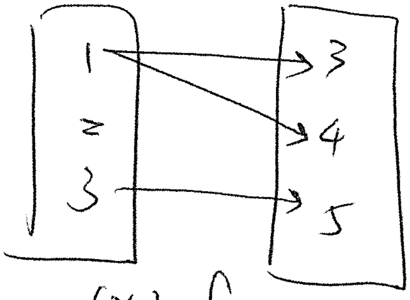
$f(\square) = \sqrt{\square} - \square^2$

$h(x) = f(x^2-1) \Rightarrow g(x) = x^2-1$.

Properties of a function

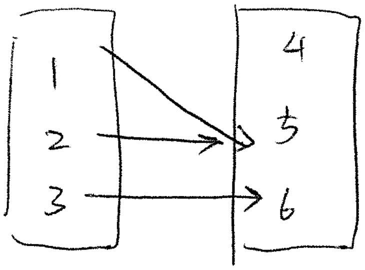


(✓) function

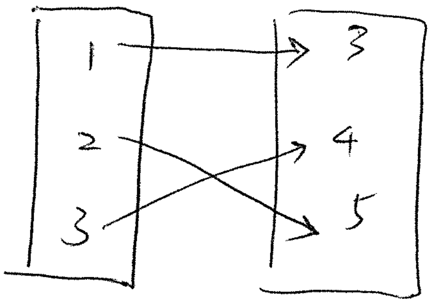


(×) function

[each element on the left can only fire an arrow].



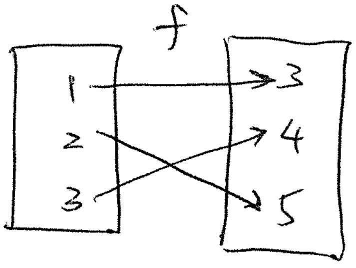
not one-to-one function

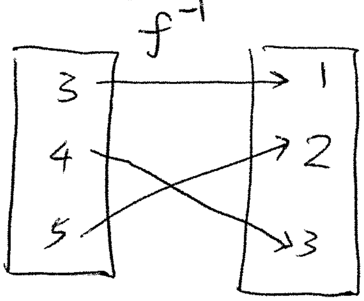


one-to-one function

[one-to-one : each element on the right can only receive an arrow]

- one-to-one function f has inverse f^{-1}

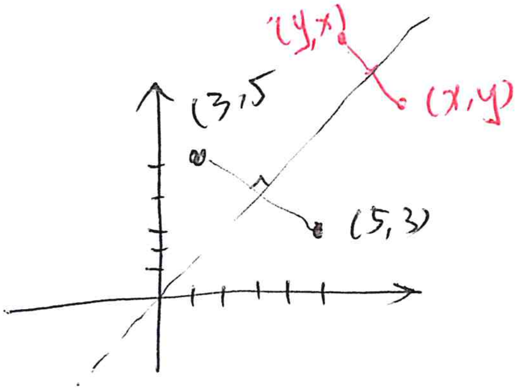


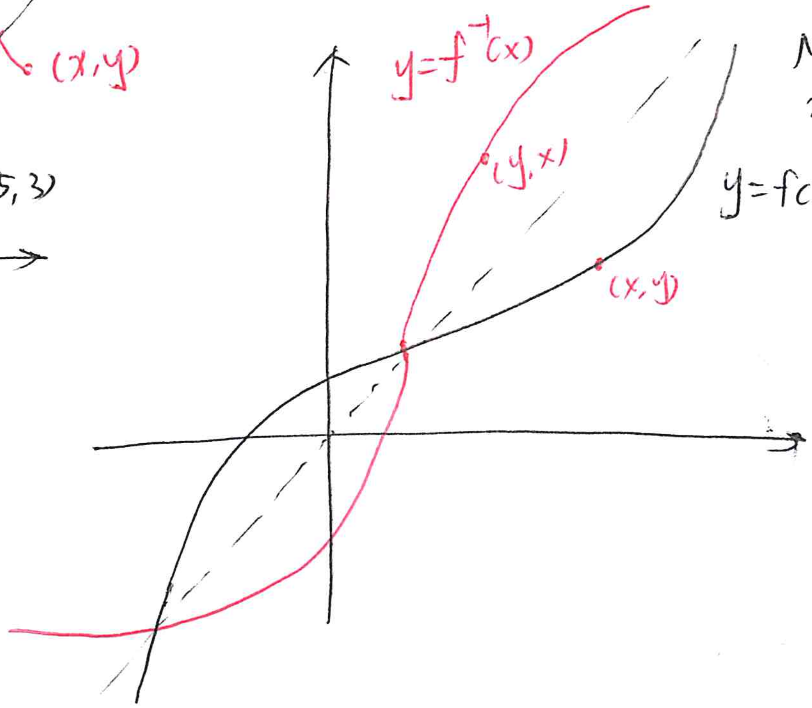


e.g. ~~f~~ $f = \{ (1,3), (2,5), (3,4)]$.

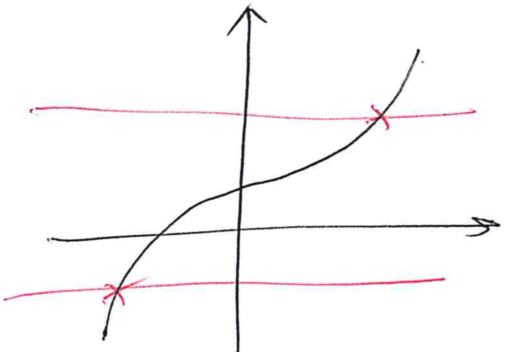
$\Rightarrow f^{-1} = \{ (3,1), (5,2), (4,3)]$

- f^{-1} is a function such that $f^{-1}(b) = a$ whenever $f(a) = b$

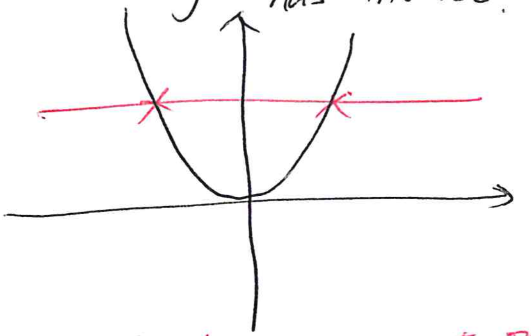




- if f passes horizontal test $\Rightarrow f$ is one-to-one
  $\Rightarrow f$ has inverse.



(√) one-to-one



(X) not one-to-one

- domain of f^{-1} = range of f.

w/4

Compute inverse

e.g. Find f^{-1} for $f(x) = 3x - 4$.

$$\begin{tikzcd} \frac{y+4}{3} = x \arrow[r, shift left, "f"] & y = 3x-4 \arrow[l, shift left, "f^{-1}"] \end{tikzcd}$$

Goal: write $y = 3x-4$ into $x = \ldots$

$$y = 3x - 4$$
$$3x - 4 = y$$
$$3x = y + 4 \quad [+4]$$
$$x = \frac{y+4}{3} \quad [\div 3]$$

So $f^{-1}(y) = \frac{y+4}{3}$; usually write $f^{-1}(x) = \frac{x+4}{3}$

- To compute inverse of $f(x)$:
  - write $y = f(x)$
  - solve $x = g(y)$, then $g = f^{-1}$

- If g is the inverse of f,

  then $g(f(x)) = x$ and $f(g(x)) = x$.

e.g. Verify $g(x) = \frac{x+4}{3}$ is the inverse of $f(x) = 3x - 4$.

- $g(f(x)) = \frac{f(x)+4}{3} = \frac{3x-4+4}{3} = x$.
- $f(g(x)) = 3 \cdot \frac{x+4}{3} - 4 = x + 4 - 4 = x$.

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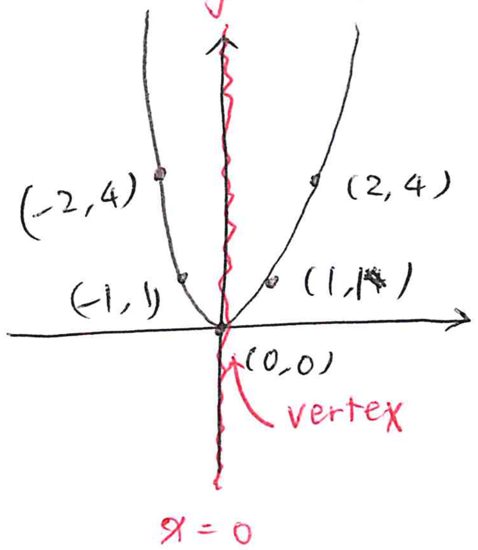
note

Chap 2. Polynomial/rational functions.

§ 2.1 Quadratic functions

- $f(x) = mx + b$, line, <u>linear function</u>
- $f(x) = ax^2 + bx + c$, parabola, <u>quadratic function</u>
  ↖ means degree 2.



- standard parabola $f(x) = x^2$
  - ■ vertex (0,0)
  - ■ open upward
  - ■ axis of symmetry $x=0$.

- vertex form: $f(x) = a(x-h)^2 + k$
  - ■ vertex (h, k)
  - ■ a is the vertical scaling factor
    - $a > 0$, openupward [$a=0 \Rightarrow f(x)=k$ is a horizontal line]
    - $a < 0$, open downward
  - ■ axis of symmetry $x=h$.

Reason:

$$y = x^2 \xrightarrow[\substack{\text{move to } (h,k) \\ \text{vertically scale by } a.}]{} \frac{y-k}{a} = (x-h)^2$$

- Fun fact :

  every quadratic function is a transformation of the standard parabola $f(x) = x^2$.

- the vertex determins

  the minimum if $a > 0$ (open upward)

  the maximum if $a < 0$ (open downward).

e.g. Find the maximum of $f(x) = -3(x-1)^2 + 2$.

vertex form: $f(x) = \underbrace{-3}_{a}(x - \underbrace{1}_{h})^2 + \underbrace{2}_{k}$

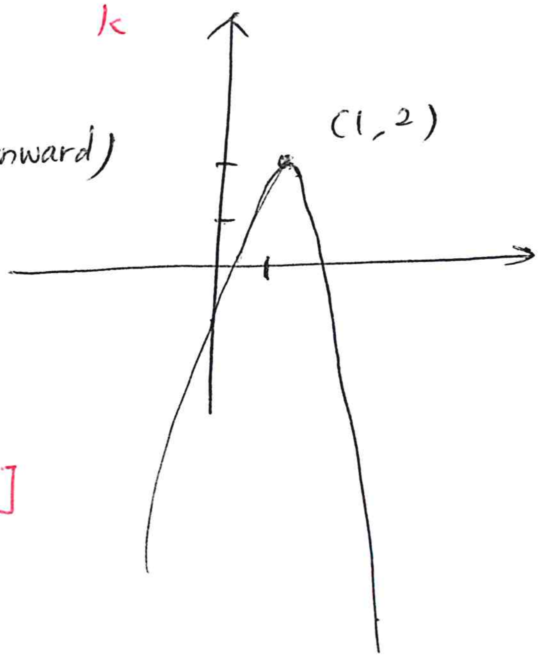
vertex $(1, 2)$

scaling factor -3 (open downward)

So maximum of f is 2,

when $x = 1$.

[This function has no minimum.]



Completing the square

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note

Goal: switching between

$$\begin{tikzcd} ax^2+bx+c \arrow[r, bend left, "\text{completing the square}"] & a(x-h)^2+k \arrow[l, bend left, "\text{expansion}"] \end{tikzcd}$$

- Observation

$$(x+q)^2 = x^2 \underbrace{+2q}x + \underbrace{q^2}$$

$$(x-q)^2 = x^2 \underbrace{-2q\,x} + \underbrace{q^2}$$

take one-half and then square it.

e.g. x^2-2x+1 is a square, $(x-1)^2$

x^2+6x+9 is a square, $(x+3)^2$

x^2-4x+4 is a square, $(x-2)^2$

e.g. Write $f(x)=x^2+6x$ in vertex form.

$$f(x) = x^2+6\underline{x+9-9} \quad [\,9=\left(\tfrac{6}{2}\right)^2\,]$$

$$=(x+3)^2-9$$

e.g. Write $f(x)=x^2-8x+4$ in vertex form.

$$f(x)=x^2-8\underline{x+16-16}+4 \quad [\,16=\left(\tfrac{-8}{2}\right)^2\,]$$

$$= \cancel{x}\,(x-4)^2-16+4 = (x-4)^2-12.$$

e.g. Write $f(x) = 2x^2 - 20x + 3$ in vertex form.

$$f(x) = 2(x^2 - 10x) + 3 \qquad [\text{combine } 2x^2 - 20x]$$

$$= 2(x^2 - 10x + 25 - 25) + 3 \qquad [\,25 = \left(\tfrac{-10}{2}\right)^2\,]$$

$$= 2\left((x-5)^2 - 25\right) + 3$$

$$= 2(x-5)^2 - 50 + 3 = 2(x-5)^2 - 47.$$

Solve quadratic equation

- The solutions of $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

e.g. Solve $x^2 + 2x - 15 = 0$.

$\sqrt{4+60} = \sqrt{64} = 8$

↓

Method 1.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot(-15)}}{2\cdot 1}$$

$$= \frac{-2 \pm \sqrt{64}}{2} = \frac{-2 \pm 8}{2} = 3, -5.$$

Method 2.

$$x^2 + 2x - 15 = x^2 + 2x + 1 - 1 - 15$$

$$= (x+1)^2 - 16$$

$(x+1)^2 - 16 = 0$

$(x+1)^2 = 16 \qquad [+16]$

$x + 1 = \pm 4 \qquad [\pm\sqrt{\ }]$

$x = -1 \pm 4 = 3, -5.$

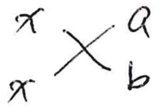
find $ab = -15$

$a + b = 2$

Method 3. $x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

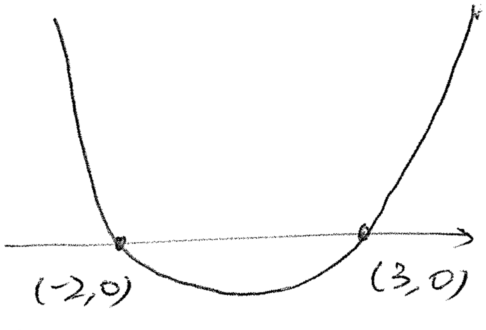
$\Rightarrow x = 3, -5$



Math 120 note

- the solutions of $f(x)=0$ give the x-intercept

e.g. Solve $x^2-x>6 \Leftrightarrow x^2-x-6>0$



① Find solution of $x^2-x-6=0$

$\rightarrow x=3,-2$

② Sketch the graph

passing $(3,0)$ and $(-2,0)$

open upward (coefficient of $x^2>0$)

③ Find points above x-axis

(>0)

Ans: $(-\infty,-2)\cup(3,\infty)$.

e.g. Solve $x^2-x\geq 6 \Leftrightarrow x^2-x-6\geq 0$.

Ans: $(-\infty,-2]\cup[3,\infty)$.

§1.2 Complex numbers.

Question: Does $x^2 = -1$ have a sdution?

Ans: No, it's not a real number.
But it has <u>imaginary solutions</u>.

- Define $i = \sqrt{-1}$, called the <u>imaginary number</u>.
  (not real)
- <u>complex number</u> means $a + bi$ for some real numbers a, b.
  a = <u>real part</u> ; b = <u>imaginary part</u>.
- rules:
  - $a+bi = c+di$ if and only if
    $a = c$ and $b = d$
    (a, b, c, d are real numbers).
  - $(a+bi) + (c+di) = (a+c) + (b+d)i$
  - $(a+bi) - (c+di) = (a-c) + (b-d)i$.
  - $(a+bi) \cdot (c+di) = ac + adi + bci + \overset{-bd}{\underset{\sim\sim}{bdii}}$
    $= (ac-bd) + (ad+bc)i$.
  - $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$
    $i^k = i^{k-4}$

e.g. $i^{100} = i^{96} = i^{92} = \cdots = i^4 = i^0 = 1$ $[100 \div 4 = 25 \ldots 0]$

$i^{2018} = i^{2014} = \cdots = i^2 = -1$ $[2018 \div 4 = 504 \ldots 2]$

conjugate:

- the **conjugate** of $a+bi$ is $a-bi$.
- fact: $(a+bi)(a-bi) = \underline{a^2+b^2}$ ↖ real number.

rationalize: multiply the conjugate of the denominator.

e.g. Rationalize $\frac{8-i}{2+i}$.

$$\frac{8-i}{2+i} = \frac{(8-i)(2-i)}{\underbrace{(2+i)(2-i)}_{\text{conjugate}}} = \frac{(16-1)+(-8-2)i}{2^2+1^2}$$

$$= \frac{15-10i}{5} = 3-2i.$$

e.g. Rationalize $\frac{1}{3+4i}$.

$$\frac{1}{3+4i} = \frac{3-4i}{\underbrace{(3+4i)(3-4i)}_{\text{conjugate}}} = \frac{3-4i}{3^2+4^2}$$

$$= \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i.$$

square root of a negative number

- square roots of $4 = \pm 2$.

  but $\sqrt{4} = 2$.

- square roots of $-4 = \pm 2i$

  but $\sqrt{-4} = 2i$.

That is, $\sqrt{-b} = \sqrt{b}\, i$ when $b>0$.

imaginary solutions

- The solutions of $ax^2+bx+c=0$ is $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

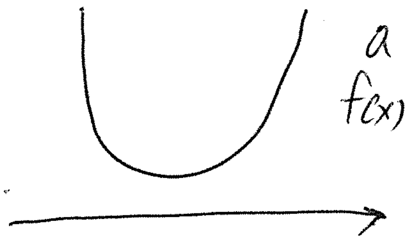
  when $b^2-4ac<0 \Rightarrow$ no real solution

  but has imaginary solution.

e.g. Solve $x^2-6x+11=0$.

$$x = \frac{6\pm\sqrt{36-44}}{2} = \frac{6\pm\sqrt{-8}}{2} = \underbrace{\frac{6\pm\sqrt{8}\,i}{2}}_{\text{imaginary}}.$$

- When $b^2-4ac<0$, the graph of $f(x)=ax^2+bx+c$ **does ~~n~~not intersect** with x-axis

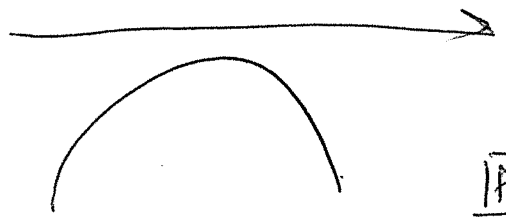


$a>0$

$f(x)$ always >0

$a<0$

$f(x)$ always <0



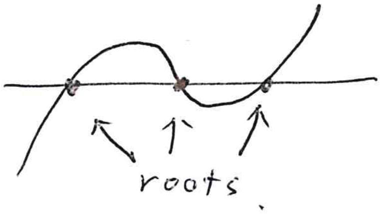
§2.3 Zeros of Polynomials
§2.4

<u>polynomial</u> : an expression

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ for some real numbers $a_n, a_{n-1}, \dots, a_0$

[all powers are integer and nonnegative]

- If $f(x)$ is a polynomial, a value c such that $f(c)=0$ is called a <u>zero</u> or a <u>root</u> of f.
- Graphically, $f(c)=0$ means $(c, 0)$ is an x-intercept.
- If $f(x) = (x-c)\cdot$(another polynomial), then c is a root.



Goal : Find all roots of a polynomial.

e.g. Write $x^3-4x^2+x+6=(x+1)$(polynomial)

Use division ~~alor~~ algorithm :

$$\begin{array}{r} x^2-5x+6 \\ x+1\overline{)\,x^3-4x^2+x+6} \\ x^3+x^2 \\ \hline -5x^2+x \\ -5x^2-5x \\ \hline 6x+6 \\ 6x+6 \\ \hline 0 \end{array}$$

product

difference

So

$x^3-4x^2+x+6=(x+1)\underline{(x^2-5x+6)}$

[Also, $x^2-5x+6=(x-2)(x-3)$

So

$x^3-4x^2+x+6=(x+1)(x-2)(x-3)$]

⇒ the roots are $-1, 2, 3$.

Division algorithm : [Keep taking out the leading term]

- $f(x)$: a polynomial, c: a real number.

Then there are unique $q(x)$ and real number r such that

$$f(x) = (x+c)\,q(x) + r$$

$q(x)$ and r are called the quotient and the remainder of $f(x) \div (x+c)$.

e.g. ~~/~~ Write $x^3+x^2+3x+2 = (x+2)\cdot q(x) + r$ with $q(x)$ a polynomial and r a real number.

Use division algorithm.

$$\begin{array}{r|l} & x^2 - x + 5 \\ \hline x+2 & x^3 + x^2 + 3x + 2 \\ & x^3 + 2x^2 \\ \hline & -x^2 + 3x \\ & -x^2 - 2x \\ \hline & 5x + 2 \\ & 5x + 10 \\ \hline & -8 \end{array}$$

So

$$x^3+x^2+3x+2 = (x+2)\underbrace{(x^2-x+5)}_{\text{quotient}}\underbrace{-8}_{\text{remainder}}$$

Note : -2 is not a root.

Remainder Theorem

If $f(x) = (x-c)\cdot q(x) + r$, then $f(c) = r$.

e.g. If $x^3+x^2+3x+2 = (x+2)\cdot q(x) + r$, find r.

Ans: $r = f(-2) = (-2)^3 + (-2)^2 + 3(-2) + 2 = -8+4-6+2 = -8$.

Factor Theorem

$$f(c) = 0 \iff f(x) = (x-c)\cdot q(x).$$

e.g. Find all roots of $x^3-2x^2-11x+12$.

- "Guess" $x=1$ is a root.
- Check $1^3-2\cdot1^2-11\cdot1+12 = 1-2-11+12 = 0 \Rightarrow x-1$ is a factor!
- Find the quotient:

$$x^3-2x^2-11x+12=(x-1)(x^2-x-12).$$

$$\begin{array}{r|l} & x^2-x-12 \\ \hline x-1 & x^3-2x^2-11x+12 \\ & x^3-x^2 \\ \hline & -x^2-11x \\ & -x^2+x \\ \hline & -12x+12 \\ & -12x+12 \\ \hline & 0. \end{array}$$

Then factorize

$$x^2-x-12 = (x-4)(x+3).$$

$$\Rightarrow x^3-2x^2-11x+12=(x-1)(x+3)(x-4).$$

Ans: roots are $1, -3, 4$.

Rational Root Theorem

If $f(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ is a polynomial with integer coefficients, then any rational root c of $f(x)$ can be written as $\frac{p}{q}$ such that $q \mid a_n$ and $p \mid a_0$

means a_0 is a multiple of p.

e.g. Find all roots of $f(x)=x^3+4x^2+7x+6$.

- Guess possible roots: $\frac{p}{q}$, $q \mid 1$, $p \mid 6$.
  $\Rightarrow q=\pm1$, $p=\pm1, \pm2, \pm3, \pm6$.
  possible rational roots: $\pm1, \pm2, \pm3, \pm6$.
- check $f(-2)=0$, so $f(x)=(x+2)\cdot q(x)$.
- Find the quotient $f(x)=(x+2)(x^2+2x+3)$
- The roots of x^2+2x+3 are $\frac{-2\pm\sqrt{4-12}}{2}=\frac{-2\pm\sqrt{-8}}{2}=\frac{-2\pm\sqrt{8}\,i}{2}$
- all roots are $-2, \frac{-2\pm\sqrt{8}\,i}{2}$

n-root Theorem

Any polynomial $f(x) = a_n x^n + a_{n-1} x^{n} + \dots + a_1 x + a_0$ $(a_n \neq 0)$.

Can bewritten as

$$f(x) = (x - \underline{c_1})(x - \underline{c_2}) \cdots (x - \underline{c_n}) \longrightarrow$$ $c_1, c_2, \dots, c_n$ are roots.

That is, every polynomial of degree n has n roots

not necessarily all real
that's why we need complex numbers.

[Same root can appear several times.
If $(x-c)^k$ is a factor of $f(x)$,
then k is the multiplicity of c.]

e.g.
$$x^3 - 4x^2 + x + 6 = (x+1)(x-2)(x-3)$$
$$x^3 - 2x^2 - 11x + 12 = (x-1)(x+3)(x-4)$$
$$x^3 + 4x^2 + 7x + 6 = (x+2)\left(x - \left(\frac{-2+\sqrt{8}\,i}{2}\right)\right)\left(x - \left(\frac{-2-\sqrt{8}\,i}{2}\right)\right)$$
$$x^2 + 2x + 1 = (x+\underline{1})(x+\underline{1})$$

-1 has multiplicity 2.

Conjugate pair

- ~~if~~ $a+bi$ is a root $\Longleftrightarrow$ $a-bi$ is a root.
- imaginary roots always come in pair, ~~one~~ which are from a quadratic equation.

e.g. Find a polynomial with root $2+3i$.

$$x = 2+3i$$
$$x - 2 = 3i$$
$$(x-2)^2 = -9$$
$$(x-2)^2 + 9 = 0$$

$\Rightarrow$ $2+3i$, $2-3i$ are ~~is~~ a root of $(x-2)^2+9$
#$(x^2 - 4x + 13)$

A summary of properties of roots.

- A polynomial of degree n has exactly n roots (real and imaginary).
- conjugate pair $a \pm bi$ always comes ~~to get~~ together.
- rational roots follow the rational root theorem.
- Use division algorithm to find the next root.

$\boxed{\text{Back to real numbers}}$

Q: How many positive roots and how many negative roots?

e.g. x^3+2x^2+3x+4 has no positive root.

$\oplus^3+2\oplus^2+3\oplus+4$ is always positive.

Similarly $-x^3-2x^2-3x-4$ has no positive root.

Fact: If the signs of ~~a polynomial~~ the coefficients of a polynomial are the same, then no positive root.

~~Rule~~ Descartes's Rule of Signs:

- Ignoring zero coefficients, # of positive roots = # sign changes − even number.

e.g. x^3+2x^2+3x+4 : no sign change ⇒ no positive root

$x^3-2x+3x+4$: 2 sign changes ⇒ 2 or 0 positive roots.

x^2-x-12 : 1 sign change ⇒ 1 positive root.

$(x-4)(x+3)$

root $= 4, -3$

e.g. $f(x) = x^3 + 2x^2 + 3x + 4$.

Then

$$f(-x) = (-x)^3 + 2(-x)^2 + 3(-x) + 4$$

$$= -x^3 + 2x^2 - 3x + 4$$

[Only odd-power terms will change signs]

~~• c is a root of f(x)~~

- c is a negative root of f(x)

$$\Updownarrow$$

$-c$ is a positive root of $f(-x)$.

- # of negative roots of f(x) = # of positive roots of f(-x).

e.g. Find the possible number of negative roots of $f(x) = x^3 + 2x^2 + 3x + 4$.

- Compute $f(-x) = \underline{-x^3 + 2x^2 - 3x + 4}$.
- 3 sign changes ⇒ $f(-x)$ has 3 or 1 positive roots.

⇒ $f(x)$ has 3 or 1 negative roots.

Math 120
note

§ 2.5 Miscellaneous equations.

square, square and square

[put a squart root on one side, the square both sides]

e.g. $\sqrt{x} + 2 = x$

$$\left[(x+q)^2 = x^2 + 2qx + q^2 \right]$$

$$\sqrt{x} = x - 2$$

$$x = (x-2)^2 = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0 \Rightarrow (x-4)(x-1) = 0$$

$$\Rightarrow x = 1, 4.$$

Check: $x = 1$

$\sqrt{x} + 2 = \sqrt{1} + 2 = 3$

$x = 1$ $(\times)$

$x = 4$

$\sqrt{x} + 2 = \sqrt{4} + 2 = 4$

$x = 4$ $(\checkmark)$

Ans: $x = 4$

e.g. $\sqrt{2x+1} - \sqrt{x} = 1$

$$\left[(\sqrt{x}+1)^2 = (\sqrt{x}+1)(\sqrt{x}+1) = x + 2\sqrt{x} + 1 \right]$$

$$\sqrt{2x+1} = \sqrt{x} + 1$$

$$2x + 1 = x + 2\sqrt{x} + 1$$

$$x = 2\sqrt{x}$$

$$x^2 = 4x \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0.$$

$x = 0$ or 4.

Check: $x = 0$

$\sqrt{2x+1} - \sqrt{x} = \sqrt{1} + \sqrt{0} = 1$ $(\checkmark)$

$x = 4$

$\sqrt{2x+1} - \sqrt{x} = \sqrt{9} - \sqrt{4} = 1$ $(\checkmark)$

Ans: $x = 0$ or 4.

powers:

- $(x^2)^3 = x^2 \cdot x^2 \cdot x^2 = x^6$

  so $(x^a)^b = x^{a \cdot b}$

- $x^{-1} = \frac{1}{x}$, $x^{\frac{1}{3}} = \sqrt[3]{x}$.

  same

- so $x^{\frac{4}{3}} = (x^4)^{\frac{1}{3}}$ or $(x^{\frac{1}{3}})^4 = \sqrt[3]{x^4}$ or $(\sqrt[3]{x})^4$

- always change to something understandable.

- when taking the root of even order, consider ±.

e.g. $x^{\frac{4}{3}} = 625$ $\Longrightarrow$ $(x^{\frac{4}{3}})^{\frac{3}{4}} = \pm \cancel{625}\, 625^{\frac{3}{4}}$

$x = \pm 625^{\frac{3}{4}}$

$\cancel{(x^{\frac{4}{3}})^3 = 625^3 \quad [\ \square^3\ ]}$

$\| \qquad \|$

$\cancel{x^4 \quad 625^3}$

$\Rightarrow x = \pm(625^3)^{\frac{1}{4}} = \pm(625^{\frac{1}{4}})^3 = \pm 5^3 = \pm 125.$

e.g. $(y-2)^{-\frac{5}{2}} = 32.$

$\frac{1}{(y-2)^{\frac{5}{2}}} = 32$

$(y-2)^{\frac{5}{2}} = \frac{1}{32}$

$y-2 = \left(\frac{1}{32}\right)^{\frac{2}{5}} = \frac{1}{32^{\frac{2}{5}}} = \frac{1}{(\sqrt[5]{32})^2} = \frac{1}{4}.$

$y = 2 + \frac{1}{4} = 2\frac{1}{4}$ or $\frac{9}{4}$.

quadratic type : [solve the quadratic equation first]

e.g. $x^4 - 14x^2 + 45 = 0$

$(x^2)^2 - 14(x^2) + 45 = 0$ [Think of $y^2 - 14y + 45 = 0$]

$\Rightarrow x^2 = 9, 5$ by quadratic formula.

$x = \pm 3, \pm\sqrt{5}$.

e.g. $x^{\frac{2}{3}} - 9x^{\frac{1}{3}} + 8 = 0$

$(x^{\frac{1}{3}})^2 - 9(x^{\frac{1}{3}}) + 8 = 0$

$\Rightarrow x^{\frac{1}{3}} = 8, 1 \Rightarrow x = 512, 1$.

---

absolute value equations : [either plus or minus, then check answers]

e.g. $|x^2 - 6| = 5x$.

$\Rightarrow$ solve $x^2 - 6 = 5x$ and $-(x^2 - 6) = 5x$.

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x = 6, -1$

$-x^2 - 5x + 6 = 0$

$x^2 + 5x - 6 = 0$

~~$x+2$~~ $(x+6)(x-1) = 0$

$x = -6, 1$

check :

$x = 6$: $|x^2 - 6| = 30$, $5x = 30$ (✓)

$x = -1$: $|x^2 - 6| = 5$, $5x = -5$ (X)

$x = -6$: $|x^2 - 6| = 30$, $5x = -30$ (X)

$x = 1$: $|x^2 - 6| = 5$, $5x = 5$ (✓)

Ans: $x = 6, 1$.

e.g. $|a-1| = |2a-3|$ $\Rightarrow$ solve ① $a - 1 = 2a - 3$ ② $a - 1 = -(2a-3)$

③ $-(a-1) = 2a - 3$ ④ $-(a-1) = -(2a-3)$

But ①,④ equivalent and ②,③ equivalent.

①,④ $\Rightarrow a = 2$, ②,③ $\Rightarrow a = \frac{4}{3}$.

Check both are answers $\Rightarrow a = 2, \frac{4}{3}$

§ 2.6 2.7 The graphs.

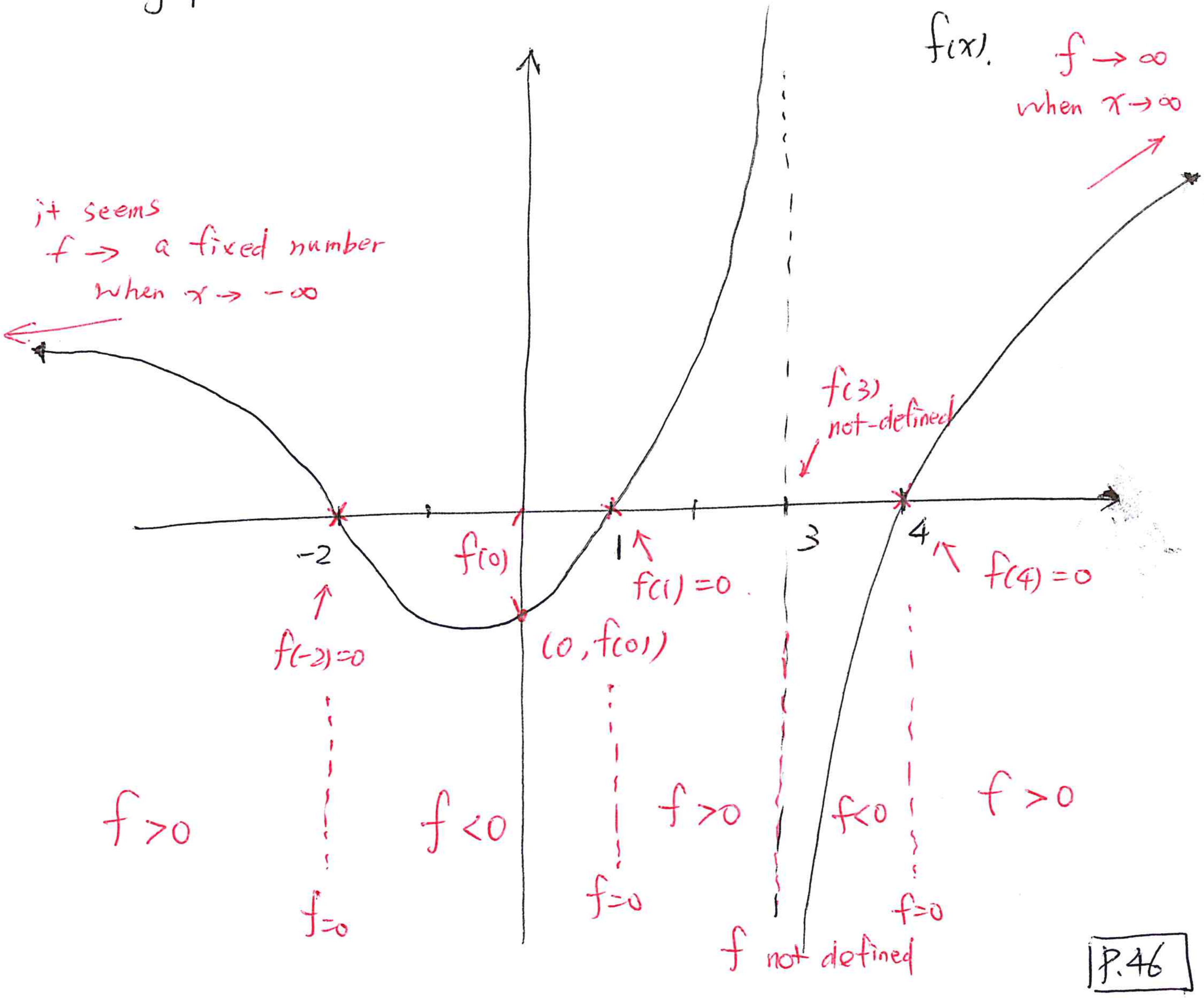
Math 120 note.

features on graphs

- y-intercept : $(0, f(0))$
- x-intercept : $(b, 0)$ with $f(b)=0$.
- some x not-defined (e.g. denominator $=0$).
- above x-axis: $f>0$; below x-axis: $f<0$.

[Other features in Calculus I :
increasing ↗ or decreasing ↘ , local ~~ex~~minimum ∨ or local maximum ∩]

- asymptotic behavior (when $x \to \infty$ or $-\infty$)



polynomial $f(x)$

- y- intercept $(0, f(0))$
- x- intercept $(b, 0)$ with $f(b)=0$.

  That is, all the roots.

- a polynomial is defined everywhere.
- use one-point test to see $f>0$ or $f<0$.

  [If $f(a)=0$, $f(b)=0$ and there is no other root between a, b]

  then $f(c)<0 \Rightarrow f<0$ on (a,b) ; c is one point on (a,b)

  $f(c)>0 \Rightarrow f>0$ on (a,b)

- asymptotic behavior : say $f(x) = a_n x^n + \dots$

  If $a_n > 0$, then $f \to \infty$ when $x \to \infty$

  $\begin{cases} f \to \infty \text{ when } x \to -\infty & \text{if } n \text{ even} \\ f \to -\infty \text{ when } x \to -\infty & \text{if } n \text{ odd.} \end{cases}$

  If $a_n < 0$, then $f \to -\infty$ when $x \to \infty$.

  $\begin{cases} f \to -\infty \text{ when } x \to -\infty & \text{if } n \text{ even} \\ f \to \infty \text{ when } x \to -\infty & \text{if } n \text{ odd.} \end{cases}$

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note

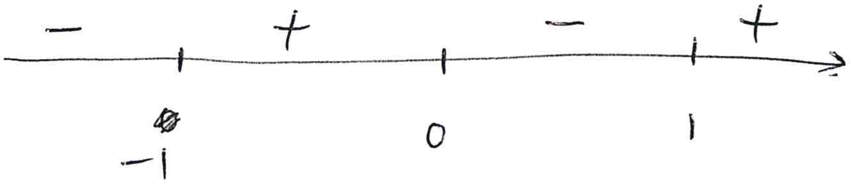
e.g. Plot $f(x) = x^3 - x$.

- y-intercept : $f(0) = 0 \Rightarrow (0,0)$
- x-intercept : $f(x) = x(x^2-1) = x(x+1)(x-1)$

  roots are $0, 1, -1$ [Rely on previous sections to solve for roots]

  $\Rightarrow (0,0), (1,0), (-1,0)$

- sign chart



$f(-2) < 0 \Rightarrow f < 0$ on $(-\infty, -1)$

$f(-0.5) > 0 \Rightarrow f > 0$ on $(-1, 0)$

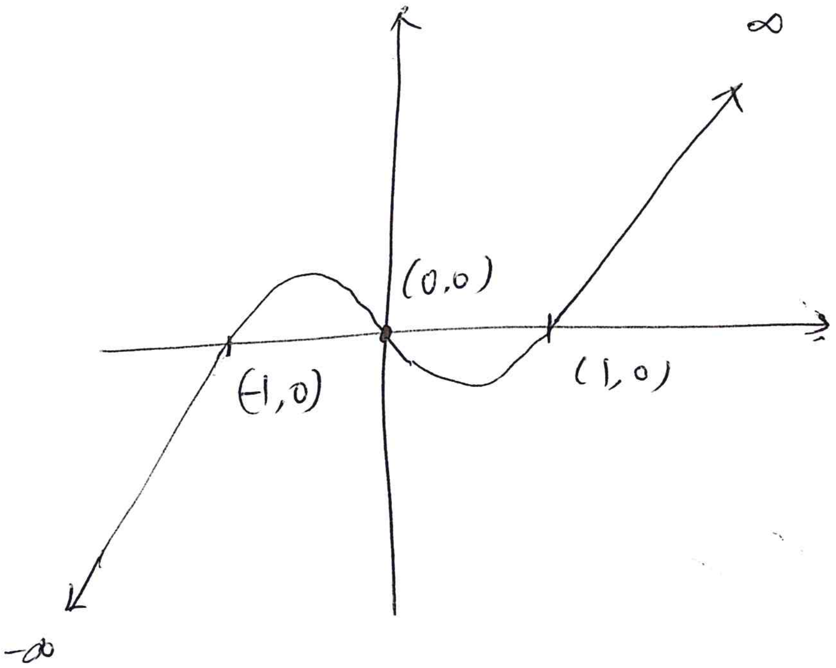
$f(0.5) < 0 \Rightarrow f < 0$ on $(0, 1)$

$f(2) > 0 \Rightarrow f > 0$ on $(1, \infty)$

- asymptotic behavior : $a_n = 1 > 0$, $n = 3$ odd.

  $\Rightarrow f \to \infty$ when $x \to \infty$

  $f \to -\infty$ when $x \to -\infty$



[You may plot more points to make it more precise.]

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note

Rational functions : $f(x) = \dfrac{p(x)}{q(x)}$, p, q polynomials

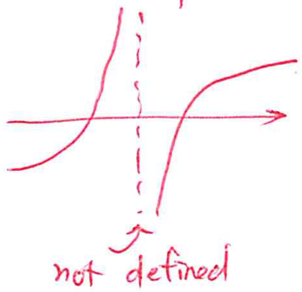
- y-intercept : $(0, f(0))$
- x-intercept : $(b, 0)$ with $f(b) = 0 \Leftrightarrow p(b) = 0$.

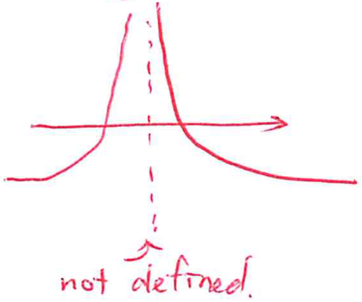
  That is, roots of $p(x)$.
- not defined : b with $q(b) = 0$.

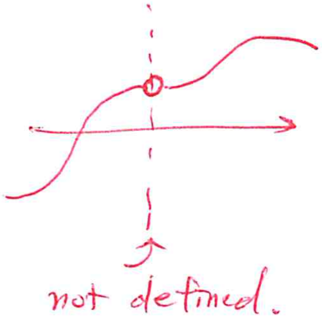
  That is, roots of $q(x)$.

  [plot some points nearby b to see the behavior]

  e.g.







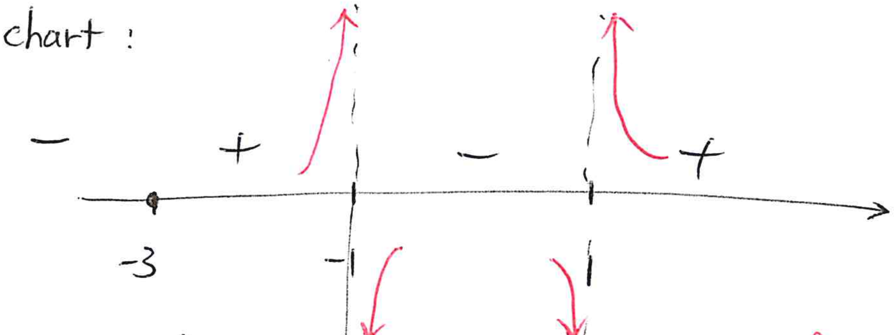
- one-point test : between (roots of p or roots of q) pick a point to decide the sign.
- asymptotic behavior : $f(x) = \dfrac{a_n x^n + \cdots}{b_m x^m + \cdots}$

When $x \to \infty$,
$$\begin{cases} f(x) \to \infty & \text{if } n > m \\ f(x) \to \dfrac{a_n}{b_m} & \text{if } n = m \\ f(x) \to 0 & \text{if } n < m \end{cases}$$

For $x \to -\infty$, consider $f(-x)$ and apply the criteria above.

e.g. Plot $f(x) = \dfrac{x+3}{(x+1)(x-1)} = \dfrac{x+3}{x^2-1}$

- y-intercept: $f(0) = -3 \Rightarrow (0, -3)$
- x-intercept: root of $x+3 \Rightarrow x = -3$
  $(-3, 0)$.
- not defined: roots of $(x+1)(x-1) \Rightarrow x = \pm 1$.
- sign chart:



$f(-4) < 0 \Rightarrow f < 0$ on $(-\infty, -3)$

$f(-2) > 0 \Rightarrow f > 0$ on $(-3, -1)$

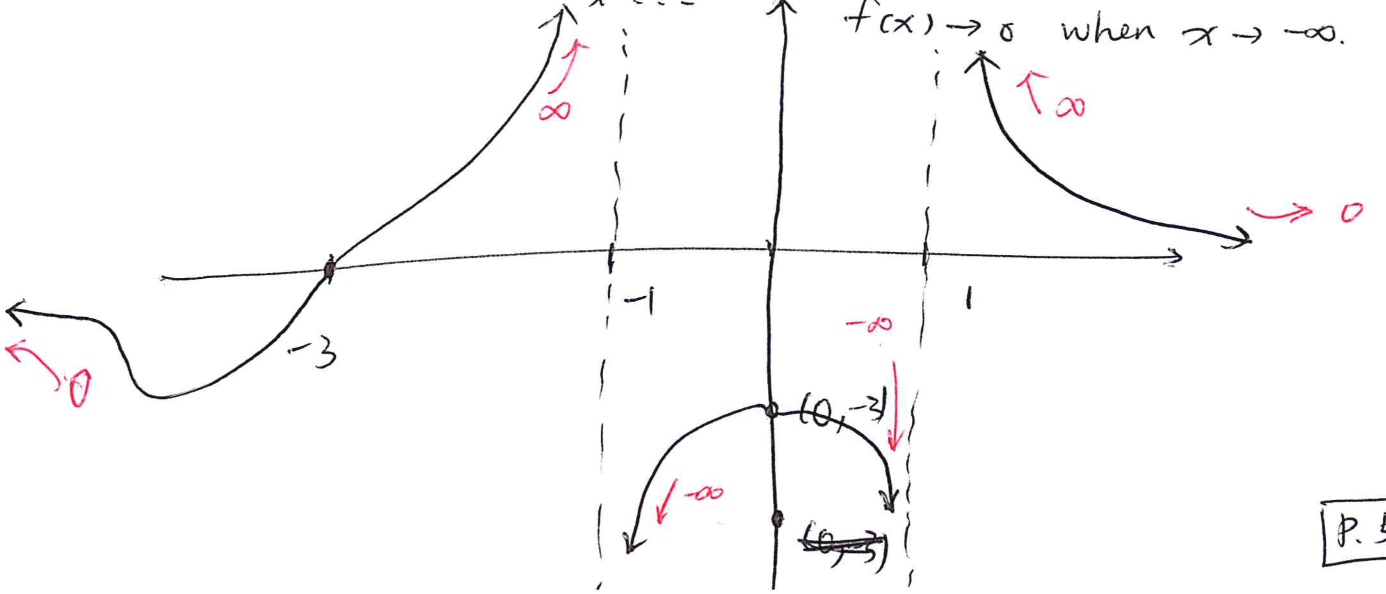
$f(0) < 0 \Rightarrow f < 0$ on $(-1, 1)$

$f(2) > 0 \Rightarrow f > 0$ on $(1, \infty)$

Try $f(-1.001)$ very large
$f(-0.009)$ very small
$f(0.009)$ very small
$f(1.001)$ very large

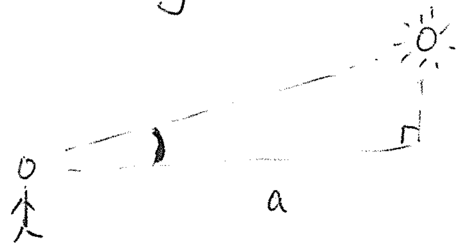
- asymptotic befavior: $\dfrac{x}{x^2} \Rightarrow \begin{matrix} n=1 \\ m=2 \end{matrix} \Rightarrow n < m$, so $f \to 0$ when $x \to \infty$

$f(-x) = \dfrac{-x \cdots}{x^2 \cdots} \Rightarrow f(-x) \to 0$ when $x \to \infty$
$f(x) \to 0$ when $x \to -\infty$.



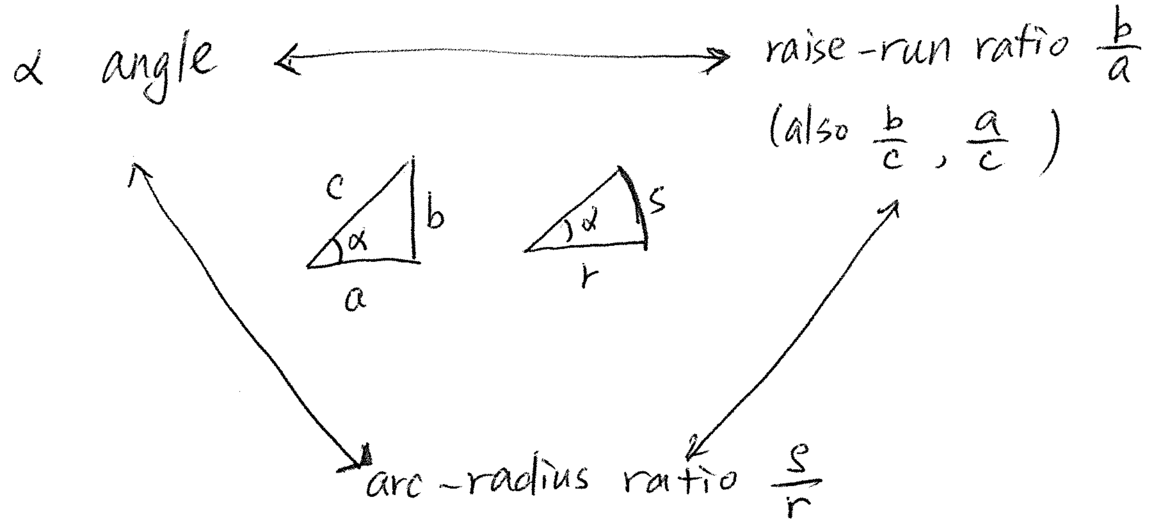
Math 120
note

Chap 3. Trigonometric functions.



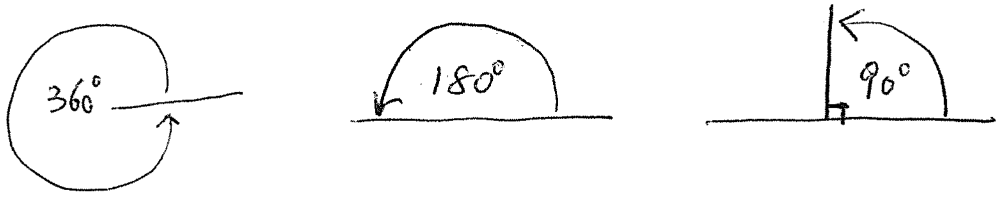
distance doesn't matter.
(or height)
the ratio matters.

angle determines the ~~ratios~~ ratios.



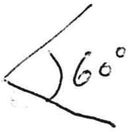
Goal: Learn how to translate one from the other.

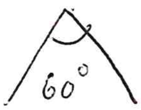
- degree notation : 360° is full circle.



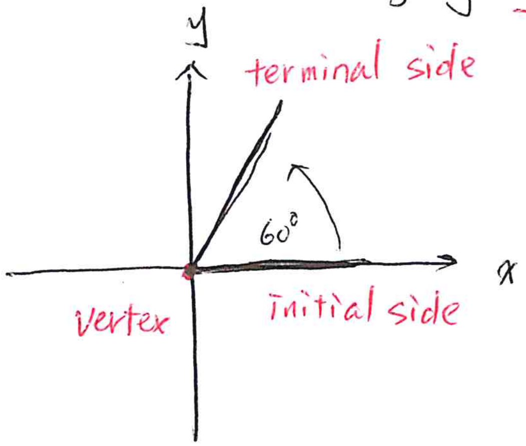
- full circle = 360°
- $1^\circ = 60'$ (60 minutes)
- $1' = 60''$ (60 seconds)

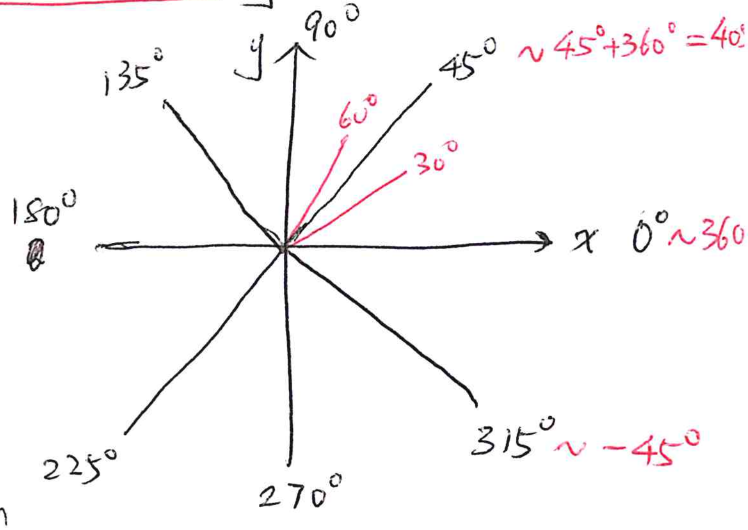
Math 120
note





- standard position : put vertex at origin
  start with initial side on positive side of x-axis
  going counterclockwisely





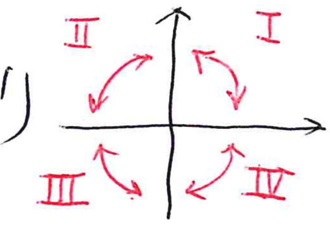
- angle can be any number from $-\infty$ to ∞
  but some have the same direction (coterminal)
- Two angles α and β are coterminal if
  $\alpha = \beta + k \cdot 360°$ for some integer k.

e.g. $45^\circ \overset{+360^\circ}{\curvearrowright} 405^\circ \overset{+360^\circ}{\curvearrowright} 765^\circ \cdots$ ~~are all coterminal~~

$\downarrow -360^\circ$

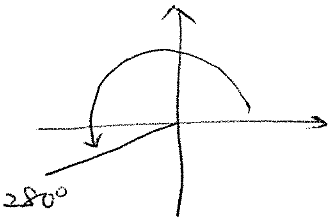
$-315^\circ \underset{-360^\circ}{\curvearrowright} -675^\circ \underset{-360^\circ}{\curvearrowright} -1035^\circ, \cdots$ } all are coterminal

- quadrants separates the directions into four region
  0°~90° : first quadrant
  90°~180° : second
  180°~270° : third
  270°~360° : fourth
  (or their coterminal angles)



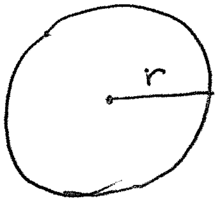
e.g. $\alpha = 1000^\circ$. Find the coterminal angle of α that is between $0^\circ \sim 360^\circ$. Then determine its quadrant.

$$1000^\circ \xrightarrow[-360^\circ]{} 640^\circ \xrightarrow[-360^\circ]{} \underline{280^\circ}$$



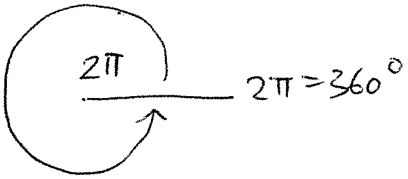
Ans: 280°, 3rd quadrant.

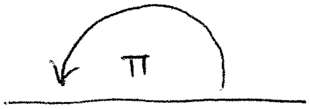
<u>radian</u>: use arc-radius ratio to describe angle.



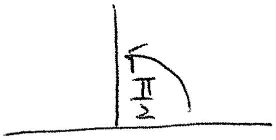
$\Rightarrow$ perimeter $= 2\pi r$.

- full circle $= \dfrac{2\pi r}{r} = 2\pi$





$\pi = 180^\circ$

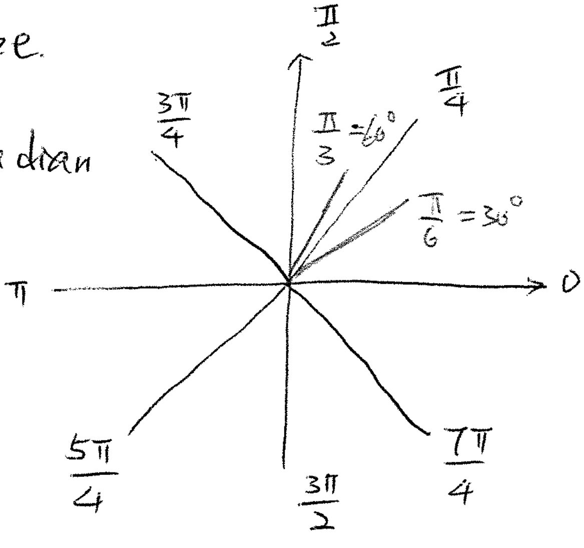


$\frac{\pi}{2} = 90^\circ$

$\frac{\pi}{3} = \cancel{15}\, 60^\circ$.

- conversion: radian $\cdot \dfrac{360^\circ}{2\pi} =$ degree.

  degree $\cdot \dfrac{2\pi}{360^\circ} =$ radian



- α and β coterminal if

  $\alpha = \beta + k \cdot 2\pi$ for some integer k.

- $0 \sim \frac{\pi}{2}$: first quadrant

  $\frac{\pi}{2} \sim \pi$: second

  $\pi \sim \frac{3\pi}{2}$: third

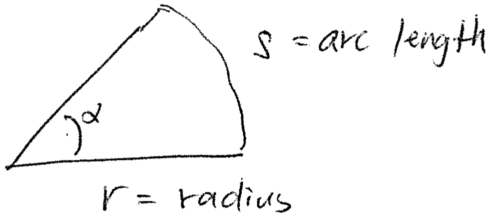
  $\frac{3\pi}{2} \sim 2\pi$: fourth

Why radian?

- radian is natural : $\frac{\text{length of arc}}{\text{length of radius}}$
  - radian is a ratio, so no unit! (You can write rad but not necessary.)

- compute the arc length.

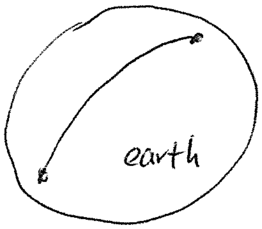
  $s = \alpha \cdot r$.



e.g. Find the ~~length~~ length of a half circle with radius 5.

angle = 180° = π
radius = 5
$\Rightarrow$ arc length = 5π

e.g. If you fly on the big circle of the earth with angle 150°. ~~How long~~
How much distance you traveled?
[suppose earth radius is 6371 km].



angle $= 150° = 150° \cdot \frac{2\pi}{360°} = \frac{5}{6}\pi \sim 2.6179\ldots$

radius = 6371 km

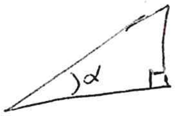
$\Rightarrow$ arc length = 6371 km × 2.6179 $\approx$ 16679 km

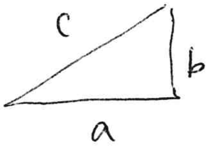
[Check out how to do conversion of degree and radian on your calculator]

§ 2.2 sin and cos

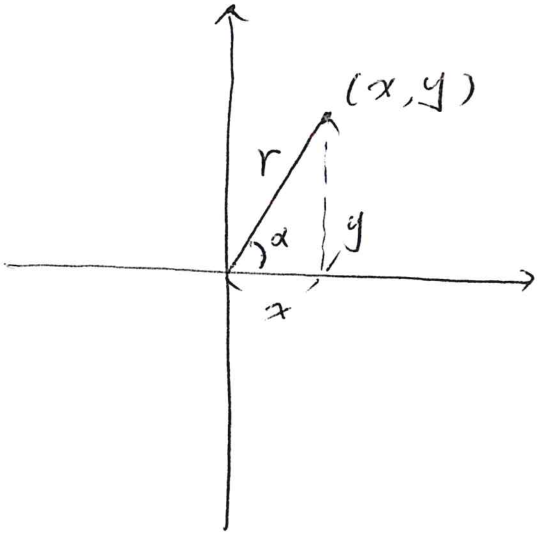
Math 120
note

Goal: translate between angle and raise-run ratio.





$$\alpha \longleftrightarrow \frac{b}{a}, \ \frac{b}{c}, \text{ or } \frac{a}{c}$$

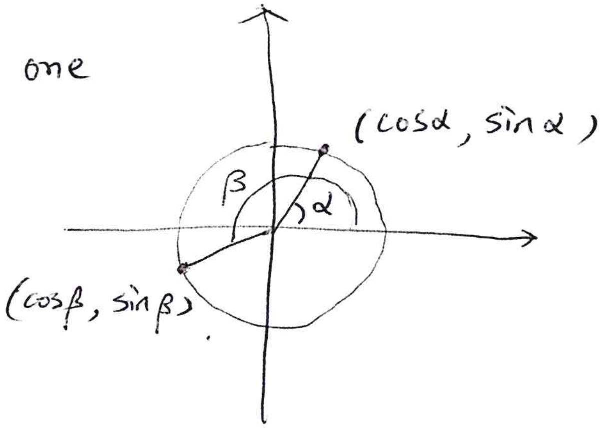


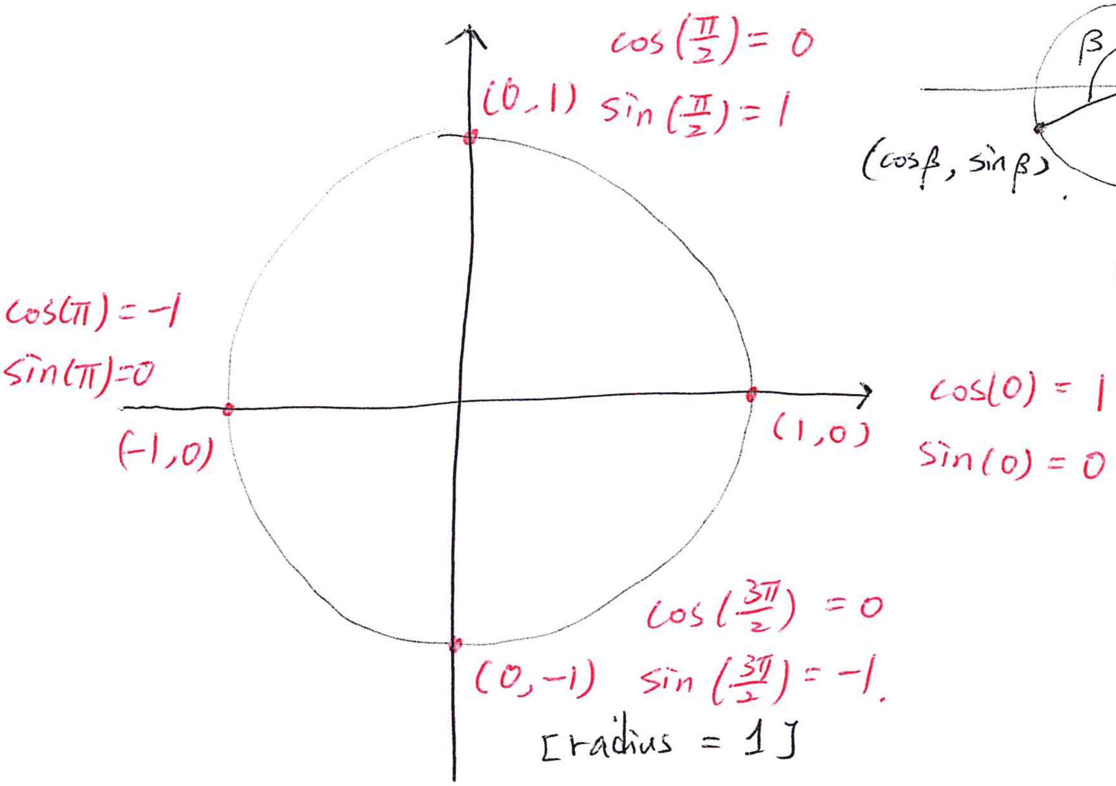
sin and cos

- sine takes angle and outputs $\frac{\text{raise}}{\text{radius}}$
- cosine takes angle and outputs $\frac{\text{run}}{\text{radius}}$

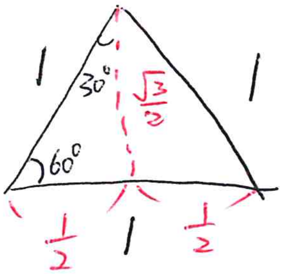
$$\begin{matrix} \sin(\alpha) = \frac{y}{r} & \xrightarrow[r=1]{\text{when}} & \sin(\alpha) = y \\ \cos(\alpha) = \frac{x}{r} & & \cos(\alpha) = x. \end{matrix}$$

- unit circle: a circle with radius one



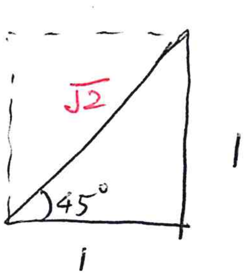


$60^\circ, 45^\circ, 30^\circ$



equilateral triangle

$$\left(\frac{1}{2}\right)^2 + h^2 = 1^2 \Rightarrow h = \frac{\sqrt{3}}{2}$$



half a square

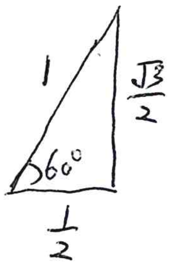
$$1^2 + 1^2 = l^2 \Rightarrow l = \sqrt{2}$$

recall: $\sin = \frac{\text{raise}}{\text{radius}}$, $\cos = \frac{\text{run}}{\text{radius}}$.

So

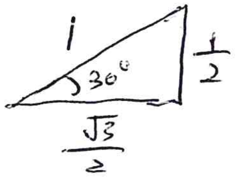
$$\sin(60^\circ) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos(60^\circ) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$



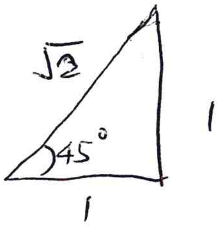
$$\sin(30^\circ) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos(30^\circ) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$



$$\sin(45^\circ) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\cos(45^\circ) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

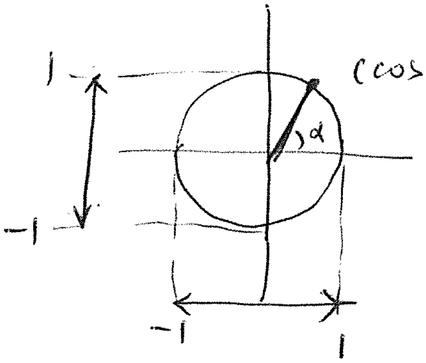


[Check out how to find ~~them~~ sin & cos on your calculator]

- sin = height of unit circle

$-1 \leq \sin\alpha \leq 1$

$-1 \leq \cos\alpha \leq 1$



- $x^2 + y^2 = 1$ on unit circle

$\Rightarrow \sin^2(\alpha) + \cos^2(\alpha) = 1$

for and α.

e.g. $\alpha = 45^\circ = \frac{\pi}{4}$

$\sin(\alpha) = \frac{1}{\sqrt{2}}$, $\cos(\alpha) = \frac{1}{\sqrt{2}}$

and $\sin^2(\alpha) + \cos^2(\alpha) = \frac{1}{2} + \frac{1}{2} = 1$.

e.g. Suppose α is in the second quadrant.

and $\sin(\alpha) = \frac{1}{3}$.

Find $\cos(\alpha)$.

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

$$\frac{1}{9} + \cos^2(\alpha) = 1 \Rightarrow \cos^2(\alpha) = \frac{8}{9}$$

$$\Rightarrow \cos(\alpha) = \pm\sqrt{\frac{8}{9}} = \pm \frac{2}{3}\sqrt{2}.$$

second quadrant $\Rightarrow \cos(\alpha) < 0$

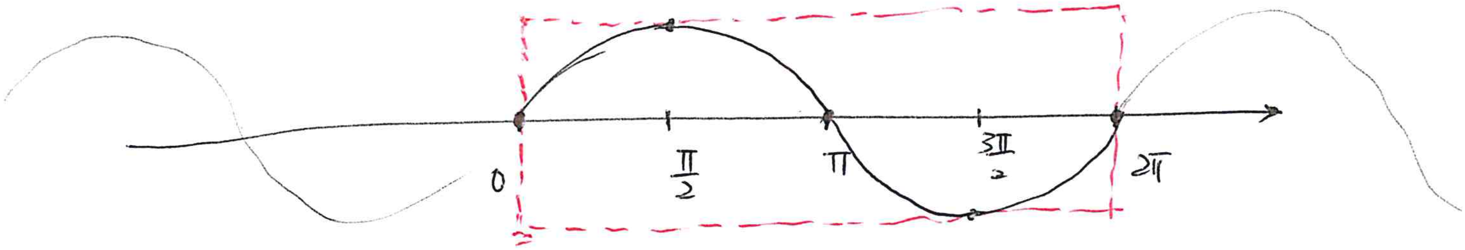
Ans: $\cos(\alpha) = -\frac{2}{3}\sqrt{2}$.

[Check desmos for the application to "simple harmonic motion."]

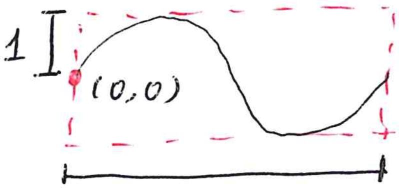
§ 3.3 Graphs of sin and cos.

sin Draw $y = \sin(x)$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
$\sin x$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0

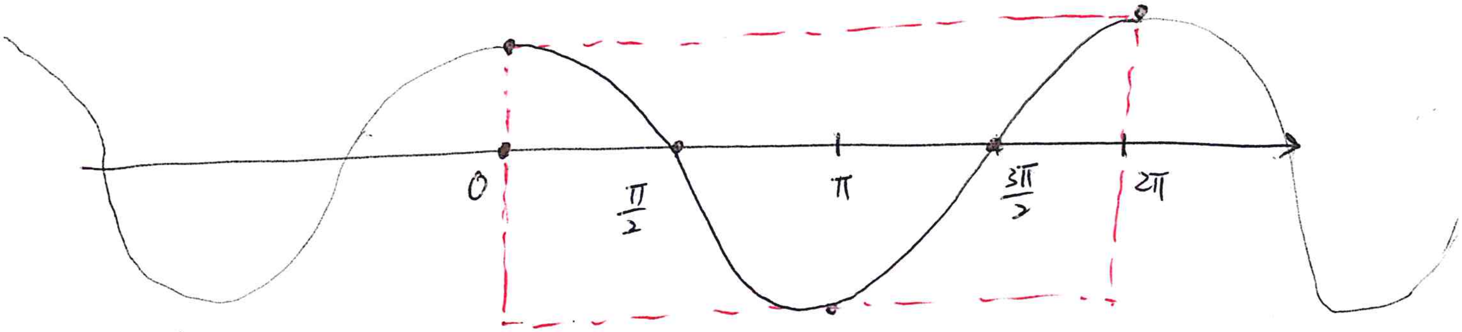


- the graph repeats the building block ⬚
- amplitude = 1, period = 2π
- reference point $(0,0)$



cos Draw $y = \cos(x)$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	2π $\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos x$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1



- different building block.
- amplitude = 1, period = 2π
- reference point $(0,0)$

[Check desmos for the graphs of sin and cos]

Fact : For the graph of sin, if I move the reference point to $(-\frac{\pi}{2}, 0)$, then it is the same as cos.

Actually, $\sin(x+\frac{\pi}{2}) = \cos(x)$

[$-\frac{\pi}{2}$ is called the phase change].

Transformations :

$$y = A\cdot\sin[B(x-C)]+D$$

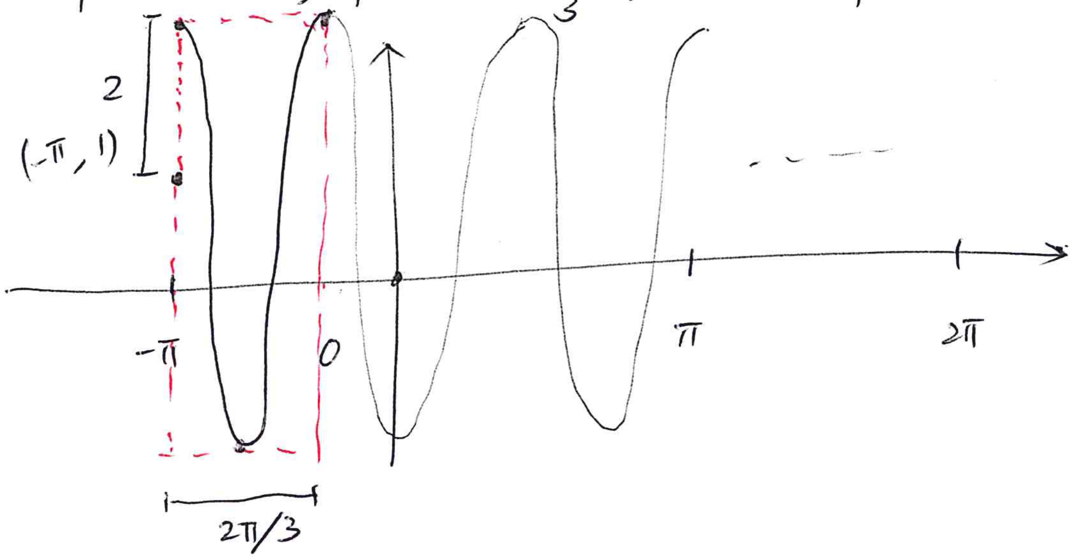
$$y = A\cdot\cos[B(x-C)]+D.$$

amplitude $= |A|$; period $= \frac{2\pi}{|B|}$

new reference point $(\underset{\substack{\uparrow \\ \text{phase change}}}{C}, \underset{\substack{\nwarrow \\ \text{vertical translation}}}{D})$

e.g. Draw $y = 2\cos(3(x+\pi))+1$.

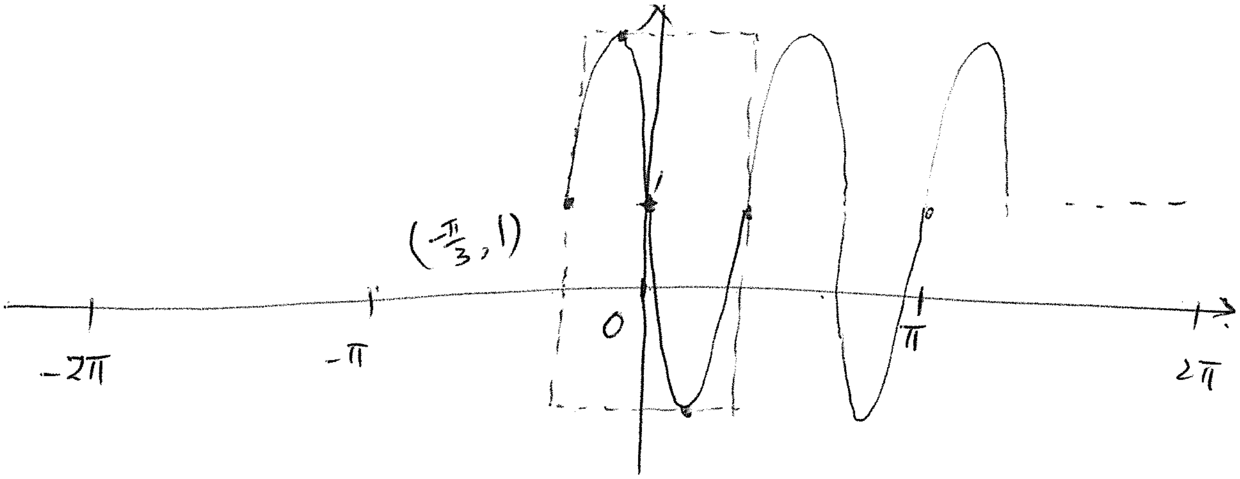
amplitude $= 2$, period $= \frac{2\pi}{3}$; reference point $(-\pi, 1)$



e.g. Draw $y = 2\sin(3x+\pi)+1$.

$= 2\sin(3(x+\frac{\pi}{3}))+1$

amplitude $=2$, period $= \frac{2\pi}{3}$, reference point $(-\frac{\pi}{3}, 1)$

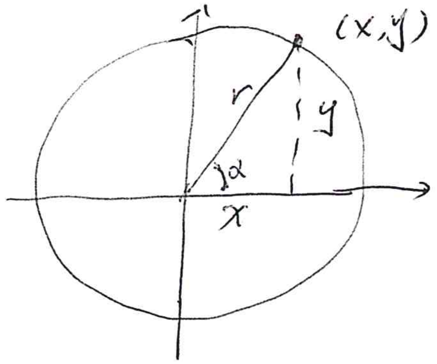


(Fourier analysis is trying to understand every waves by sin and cos.)

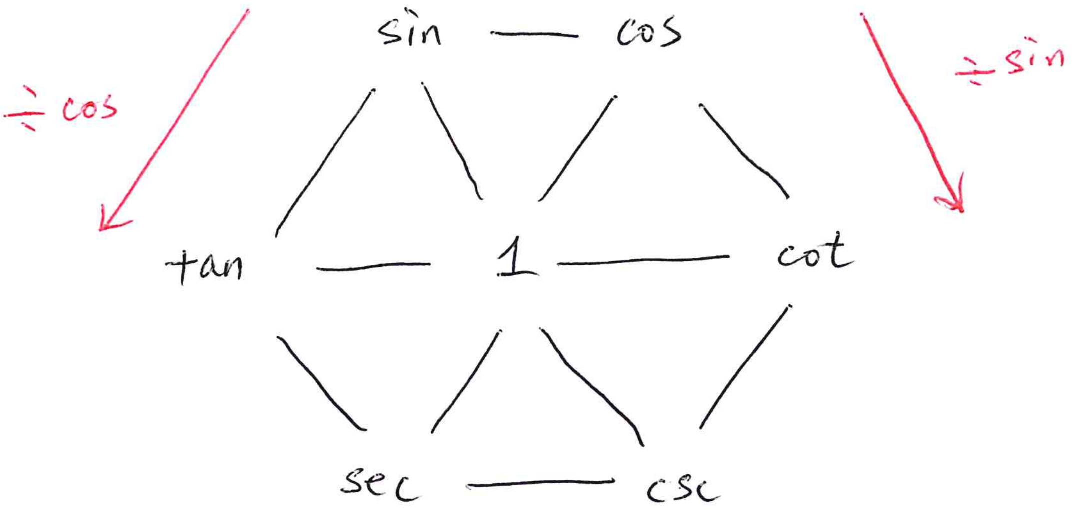
§ 3.4 Other trig functions.

Six functions:

sine	$\sin(\alpha)$		$= \cancel{\frac{x}{r}}\ \frac{y}{r}$
cosine	$\cos(\alpha)$		$= \cancel{\frac{y}{r}}\ \frac{x}{r}$
tangent	$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$		$= \frac{y}{x}$ = slope
secand	$\sec(\alpha) = \frac{1}{\cos(\alpha)}$		$= \frac{r}{x}$
co secand	$\csc(\alpha) = \frac{1}{\sin(\alpha)}$		$= \frac{r}{y}$
co tangent	$\cot(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)}$		$= \frac{x}{y}$



The hexigon



- ~~antipo~~ opposite vertex means reciprocal:

  $\csc = \frac{1}{\sin}$, $\sec = \frac{1}{\cos}$, $\cot = \frac{1}{\tan}$

- ▽ means Pythagorean identity.

  $\sin^2(x) + \cos^2(x) = 1$

  $\tan^2(x) + 1^2 = \sec^2(x)$

  $1^2 + \cot^2(x) = \csc^2(x)$.

e.g. Find $\tan(90°)$ and $\sec(150°)$ and $\csc(45°)$.

Key: change everything to sin and cos.

$$\tan(90°) = \frac{\sin(90°)}{\cos(90°)} = \frac{1}{0} \Rightarrow \text{not defined}.$$

$$\sec(150°) = \frac{1}{\cos(150°)} = \frac{1}{\left(\cancel{-\frac{1}{2}}\right)_{-\frac{\sqrt{3}}{2}}} = \cancel{-2}\ -\frac{2}{\sqrt{3}}$$

$$\csc(45°) = \frac{1}{\sin(45°)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2}.$$

Now use the desmos link "Trigonometric functions properties" on CourseSpaces.

to find out the following properties.

$\boxed{\sin}$
- zero at $0, \pi, 2\pi, \ldots$
- defined everywhere
- period $= 2\pi$

$\boxed{\cos}$
- zero at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$
- defined everywhere
- period $= 2\pi$

$\boxed{\tan}$
- zero at $0, \pi, 2\pi, \ldots$ (same as sin)
- not defined at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ (zeros of cos)
- period $= \pi$

sec
- never zero
- not defined at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ (zeros of cos)
- period = 2π

csc
- never zero
- not defined at $0, \pi, 2\pi, \ldots$ (zeros of sin)
- period = 2π

cot
- zero at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ (zeros of cos)
- not defined at $0, \pi, 2\pi, \ldots$ (zeros of sin)
- period = π

§ 3.5 Inverse functions.

$$\begin{tikzcd} \text{angle} \arrow[r, shift left, "{\begin{array}{c}\sin,\ \cos,\ \tan\\ \sec,\ \csc,\ \cot\end{array}}"] & \text{ratios} \arrow[l, shift left, "{\begin{array}{c}\boxed{\arcsin,\ \arccos,\ \arctan}\\ \text{arcsec},\ \text{arccsc},\ \text{arccot}\end{array}}"] \end{tikzcd}$$

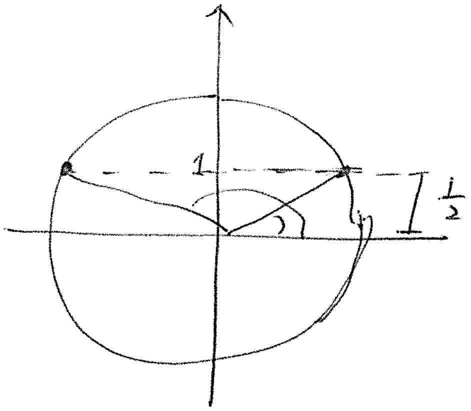
e.g.

Find angle α such that $\sin\alpha = \frac{1}{2}$.

α can be

$30°, 150°, 390°, 410°, \ldots$

$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6}+2\pi, \frac{5\pi}{6}+2\pi, \ldots$

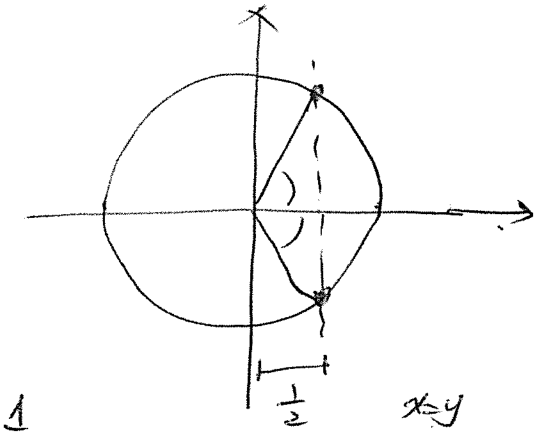


e.g. Find angle α such that $\cos\alpha = \frac{1}{2}$.

α can be

$60°, -60°, 420°, 300°, \ldots$

$\frac{\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}+2\pi, -\frac{\pi}{3}+2\pi, \ldots$

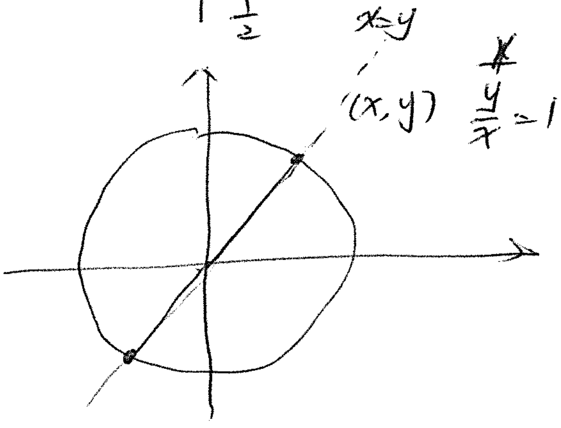


e.g. Find angle α such that $\tan\alpha = 1$

α can be

$45°, 215°, 405°, 575°, \ldots$

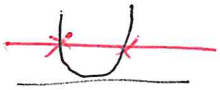
$\frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{4}+2\pi, \frac{5\pi}{4}+2\pi, \ldots$



Recall :

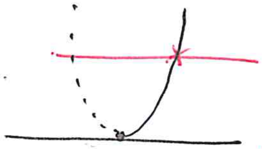
- a function has an inverse if it is one-to-one
  (pass the horizontal test)
- the inverse of a function has the graph that is the reflection along $x=y$ of the original graph.

e.g. $y=x^2$ fails the horizontal test.



(×)

~~By~~ But $y=x^2$ $(x \geq 0)$ ~~is ok~~ passes the test.



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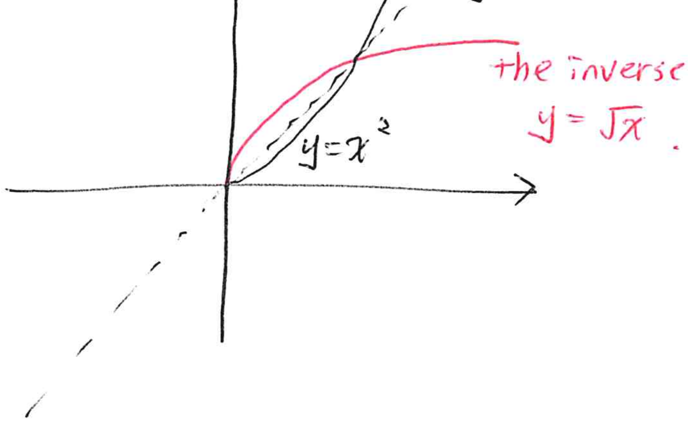


The "inverse" trig functions [See graphs on desmos]

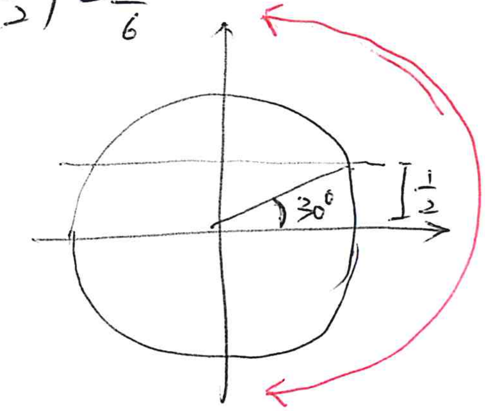
arcsin $\xleftrightarrow{\text{inverse}}$ $\sin(x)$ $\left(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right)$

arc cos $\xleftrightarrow{\text{inverse}}$ $\cos(x)$ $\left(0 \leq x \leq \pi\right)$

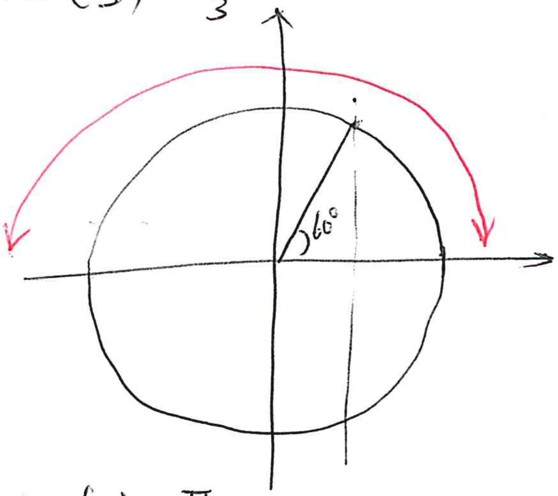
arc tan $\xleftrightarrow{\text{inverse}}$ $\tan(x)$ $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

e.g. Find $\arcsin(\frac{1}{2})$, $\arccos(\frac{1}{2})$, $\arctan(1)$.

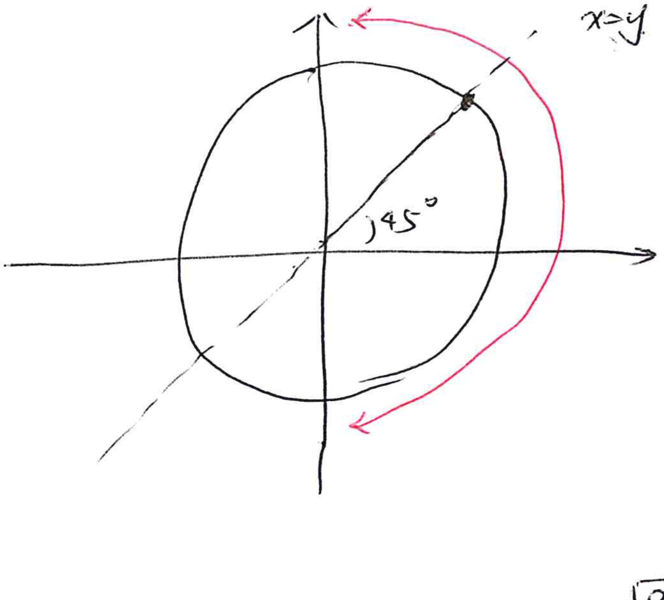
$$-\frac{\pi}{2} \le \arcsin(\tfrac{1}{2}) \le \frac{\pi}{2} \Rightarrow \arcsin(\tfrac{1}{2}) = \frac{\pi}{6}$$



$0 \le \arccos(\frac{1}{2}) \le \pi \Rightarrow$ ~~arcsin~~ $\arccos(\frac{1}{2}) = \frac{\pi}{3}$



$$-\frac{\pi}{2} < \arctan(1) < \frac{\pi}{2} \Rightarrow \arctan(1) = \frac{\pi}{4}.$$



§ 3.6 Applications.

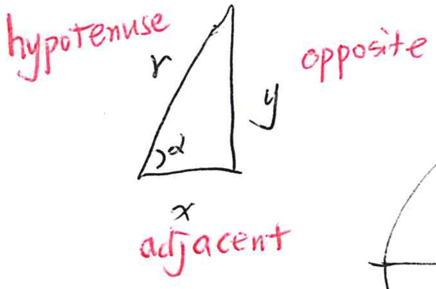
Goal: Use the right triangle directly.

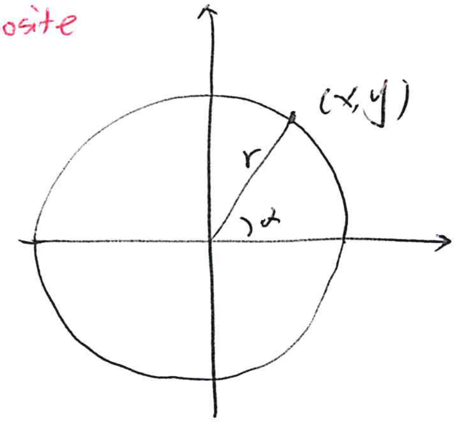
Recall:

$\sin(\alpha) = \frac{y}{r}$

$\cos(\alpha) = \frac{x}{r}$

$\tan(\alpha) = \frac{y}{x}$

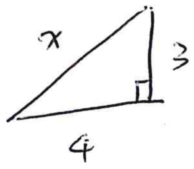




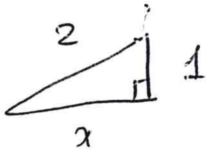
Key: side + side ⟶ the other side

side + angle ⟶ another side ⟶ all sides

Side + side

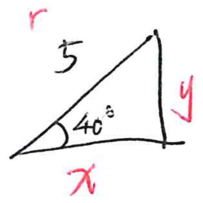


$x^2 = 3^2 + 4^2 \Rightarrow x = 5$



$1^2 + x^2 = 2^2 \Rightarrow x = \sqrt{3}$.

side + angle



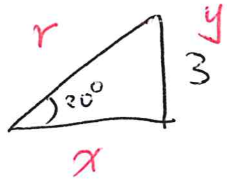
$\Rightarrow$

$x = r \cdot \frac{x}{r}$

$\underline{5 \cdot \cos(40°)}$

$y = r \cdot \frac{y}{r}$

$5 \cdot \sin(40°)$



$\Rightarrow$

$x \cdot \tan(20°) = 3$

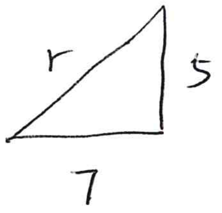
$\underline{3 \div \tan(20°)}$

$3 \div \sin(20°)$

$r \cdot \sin(20°) = 3$

find other trig functions

e.g. Suppose α is in the first quadrant. If $\tan(\alpha) = \frac{5}{7}$, find $\sin(\alpha)$ and $\cos(\alpha)$.



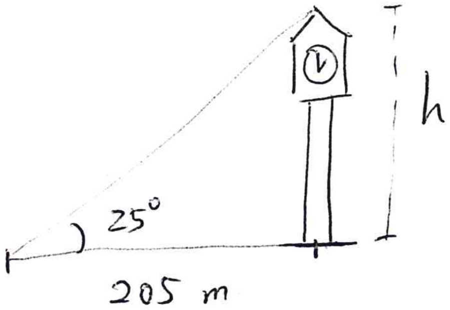
$r^2 = 5^2 + 7^2 \Rightarrow r = \sqrt{74}$

so $\sin(\alpha) = \frac{5}{\sqrt{74}}$

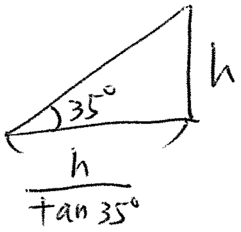
$\cos(\alpha) = \frac{7}{\sqrt{74}}$.

measure a tower

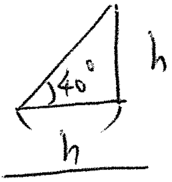
$h = 205 \cdot \tan(25°) \sim 96\ m$



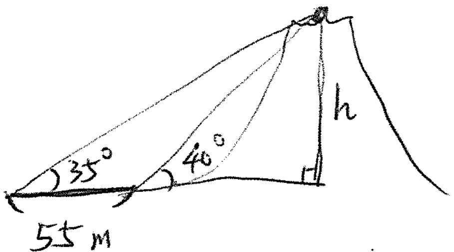
measure a mountain



$\frac{h}{\tan 35°}$



$\frac{h}{\tan 40°}$



Mount Douglas

$$\Rightarrow \frac{h}{\tan 35°} - \frac{h}{\tan 40°} = 55$$

$$h \cdot \left(\frac{1}{\tan 35°} - \frac{1}{\tan 40°}\right) = 55$$

$$h = 55 \div \left(\frac{1}{\tan 35°} - \frac{1}{\tan 40°}\right)$$

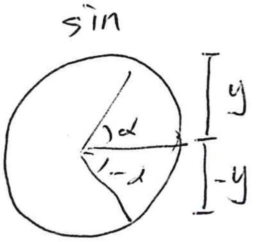
$$\approx 55 \div (1.4281 - 1.1917) \approx 233 \text{ m}.$$

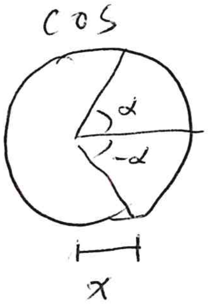
§3.7 Identities.

Math 120
note.

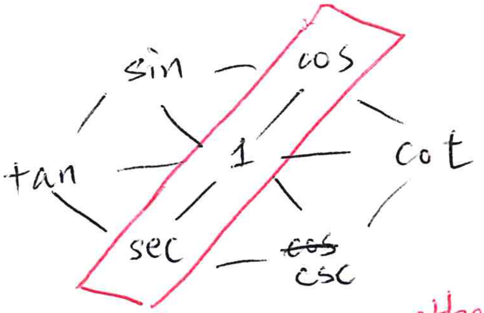
even or odd

sin is ~~even~~ odd
cos is even





$-\sin(\alpha) = \sin(-\alpha)$ $\cos(\alpha) = \cos(-\alpha)$



even functions, others are all odd functions.

sum/difference formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

e.g. Find ~~sin(α~~ $\sin(15°)$ and $\cos(15°)$

$$\sin(15°) = \sin(60° - 45°) = \sin 60° \cos 45° - \cos 60° \sin 45°$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos(15°) = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Double angle [$\alpha = \beta$]

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

$$= (\cos^2\alpha + \sin^2\alpha) - 2\sin^2\alpha = 1 - 2\sin^2\alpha$$

$$= 2\cos^2\alpha - \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$$

e.g. $\sin 90° = 1$

also $\sin 90° = 2\sin 45° \cos 45° = 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1$

2α

$\alpha = 45°$

e.g. ~~sin 30~~

$\cos(30°) = 1 - 2\sin^2(15°)$ $\Rightarrow$ $\sin^2(15°) = \frac{1-\cos(30°)}{2}$

$\frac{\sqrt{3}}{2}$ //

$= 2\cos^2(15°) - 1$ $\qquad$ $\cos^2(15°) = \frac{1+\cos(30°)}{2}$

$\sin 15° = \pm\sqrt{\frac{1-\sqrt{3}/2}{2}}$ $\qquad$ $\cos 15° = \pm\sqrt{\frac{1+\sqrt{3}/2}{2}}$

[Since 15°, both sin and cos are positive]

Half angle [Use double-angle formula in a reversed way]

$\cos(2\alpha) = 1 - 2\sin^2\alpha$ $\qquad$ $\sin\alpha = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}}$

$\Rightarrow$

$= 2\cos^2\alpha - 1$ $\qquad$ $\cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$

so $\tan\alpha = \pm\sqrt{\frac{1-\cos(2\alpha)}{1+\cos(2\alpha)}}$

Whether + or − depends on the quadrant of α.

Math 120
note

Fun fact: sin and cos can combine together.

e.g. $3 \cdot \sin x + 4 \cos x$

$= 5 \cdot \left(\frac{3}{5} \cdot \sin x + \frac{4}{5} \cos x \right)$ let $\cos\alpha = \frac{3}{5}$, $\sin\alpha = \frac{4}{5}$

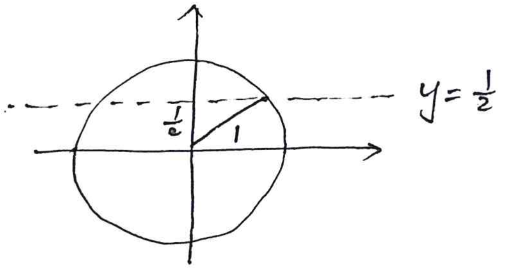
$= 5 \cdot (\cos\alpha \sin x + \sin\alpha \cos x)$

$= 5 \cdot \sin(x + \alpha)$

[Idea: Wave + Wave is another wave].

§ 3.8 Equations.

e.g. Solve $\sin(x) = \frac{1}{2}$.
$\uparrow$
the y-coordinate is $\frac{1}{2}$
at some point on the unit circle



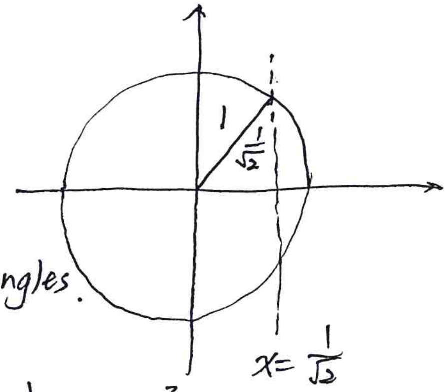
$\Rightarrow$ $x = 30°$ or $150°$ or their coterminal angles.

Ans: $x \in \{\underbrace{\frac{\pi}{6} + 2k\pi}_{\substack{\uparrow \\ \text{element}}} \mid \underbrace{k \in \mathbb{Z}}_{\substack{\uparrow \\ \text{condition}}}\} \cup \{\frac{5}{6}\pi + 2k\pi \mid k \in \mathbb{Z}\}$

e.g. $\{k\pi \mid k \in \mathbb{Z}\} = \{\ldots -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots\}$

$\{\pi + 2k\pi \mid k \in \mathbb{Z}\} = \{\ldots, -5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi, \ldots\}$

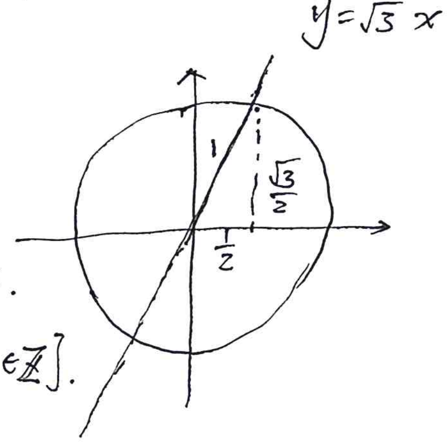
e.g. Solve $\cos(x) = \frac{1}{\sqrt{2}}$
$\uparrow$
the x-coordinate is $\frac{1}{\sqrt{2}}$



$\Rightarrow$ $x = 45°$ or $-45°$ or their coterminal angles.

Ans: $x \in \{\frac{\pi}{4} + 2\pi \cdot k \mid k \in \mathbb{Z}\} \cup \{-\frac{\pi}{4} + 2k\pi \mid k \in \mathbb{Z}\}$.

e.g. Solve $\tan(x) = \sqrt{3}$
$\uparrow$
the slope $= \sqrt{3}$.



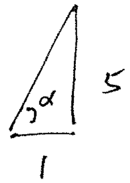
$\Rightarrow$ $x = 60°$ or $-120°$ or their coterminal angles.

Ans: $x \in \{\frac{\pi}{3} + 2k\pi \mid k \in \mathbb{Z}\} \cup \{-\frac{2}{3}\pi + 2k\pi \mid k \in \mathbb{Z}\}$.

Solve by calculator :

e.g. $\tan(\alpha)=5$ and $0 \leq \alpha \leq 90°$.

Find α.

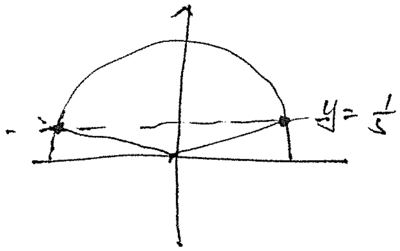


$\Rightarrow \tan^{-1}(5) = 78.69\ldots$ degree. $\Rightarrow \alpha = 78.69\ldots°$

e.g. $\sin(\alpha)=\frac{1}{3}$ and $90° \leq \alpha \leq 180°$

$\Rightarrow \sin^{-1}(\frac{1}{3}) = 19.47°$ ← not in $90° \sim 180°$

$\alpha = 180° - 19.47° = 160.53°$



!! make sure DEG or RAD.

!! make sure which quadrant.

same equation in disguise

e.g. Solve $5\sin(\alpha)=\cos(\alpha)$, $0° \leq \alpha \leq 90°$.

$\Rightarrow \quad \tan(\alpha)=\frac{1}{5} \Rightarrow \alpha = \tan^{-1}(\frac{1}{5}) = 11.31°$

e.g. $\frac{1}{2}\sin(\alpha)+\frac{\sqrt{3}}{2}\cos(\alpha)=1$. Find all α.

$$\cos(60°)\sin(\alpha) + \sin(60°)\cos(\alpha) = 1$$

$$\sin(\alpha + 60°) = 1$$

$\Rightarrow \quad \alpha + 60° = 90°$ or its coterminal angle.

$\alpha = 30°$ or its coterminal angle.

$\Rightarrow \quad \alpha \in \{\frac{\pi}{6} + 2k\pi \mid k \in \mathbb{Z}\}$ or $\alpha \in \{30° + 360k° \mid k \in \mathbb{Z}\}$.

Quadratic + trig combo

e.g. Solve $4\sin^2\alpha - 4\sin\alpha + 1 = 0$.

Let $y = \sin\alpha$. $4y^2 - 4y + 1 = 0 \Rightarrow$ ~~2y+1~~

$(2y-1)^2 = 0$

$\Rightarrow y = \frac{1}{2}$

Solve $\sin(\alpha) = \frac{1}{2}$

$\Rightarrow$ $\alpha = 30°$ or $150°$ or their coterminal angles.

$\alpha \in \left\{ \frac{\pi}{6} + 2k\pi \;\middle|\; k \in \mathbb{Z} \right\} \cup \left\{ \frac{5}{6}\pi + 2k\pi \;\middle|\; k \in \mathbb{Z} \right\}$.

Chap 4: Exp and log ~ ~ ~

Math 120
note

Suppose there are two magic boxes that generate money by the year x

	Box A	Box B	
If	x^2	2^x	when $x=10$, then $B>A$.
If	x^5	2^x	when $x=10$, then $A>B$.

But when x is very large, $2^x \gg x^5 \gg x^2$.

$2^x, 3^x, \ldots,$ or even 1.1^x are called <u>exponential growth</u>

$1, x, x^2, \ldots x^{100} \ldots$ are called <u>polynomial growth</u>

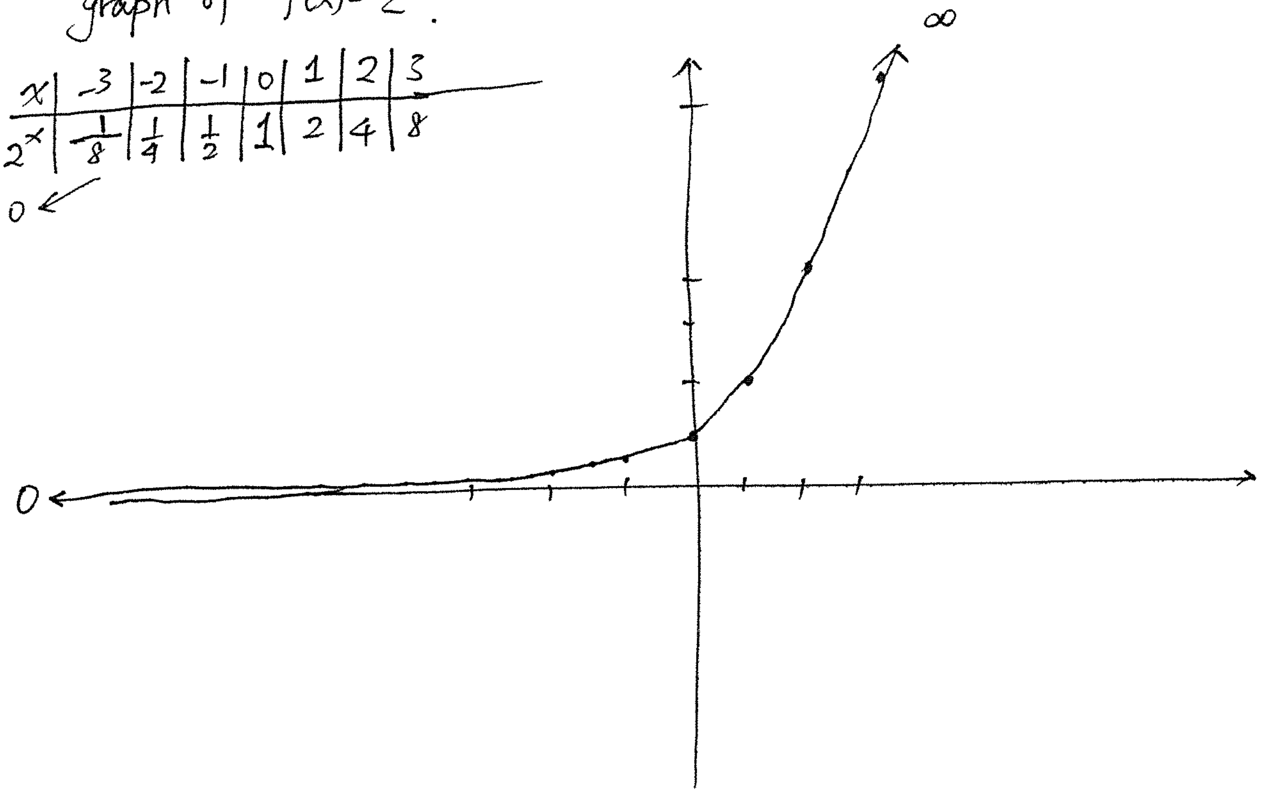
They are VERY different, and exponential growth will eventually bigger than polynomial growth.

$1.1^x > x^{100}$ when x is very large.

graph of $f(x)=2^x$.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$0 \leftarrow$



meaning of 2^x

$$2^3 = 2 \cdot 2 \cdot 2 = 8 .$$

$$2^{\frac{1}{2}} = \sqrt{2} \quad , \quad 2^{\frac{1}{3}} = \cancel{\sqrt{3}} \ \sqrt[3]{2}$$

$$2^{\frac{p}{q}} = \sqrt[q]{2^p} = \left(\sqrt[q]{2}\right)^p$$

$$2^{-r} = \frac{1}{2^r} .$$

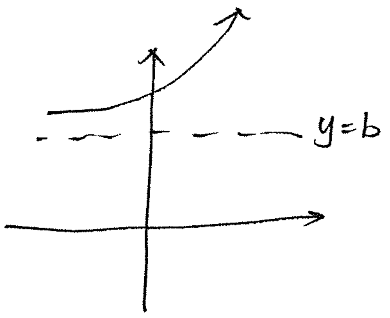
How about 2^{π}...

$$2^3 , 2^{3.1} , 2^{3.14} , 2^{3.141} , \cdots \longrightarrow 2^{\pi}$$

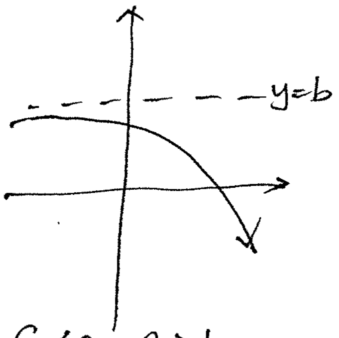
Fact : $2^{\text{whatever}} > 0$

$f(x) = c \cdot a^x + b \ , \ a > 0$, $\cancel{c \neq 0}$

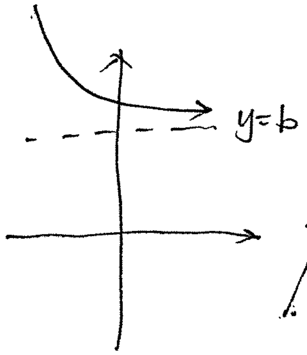
- domain = $\mathbb{R}$
- range = $\begin{cases} (b, \infty) & \text{if } c > 0 \\ (-\infty, b) & \text{if } c < 0 \end{cases}$ ~~since~~ since $a^x > 0$.



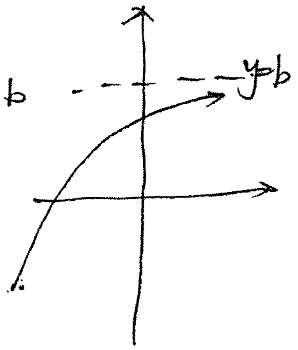
$c > 0 , a > 1$



$C < 0 , a > 1$



$C > 0 , a < 1$



$C < 0 , a < 1$

exp is like a parallel world

e.g. Solve $2^{5x+3} = 2^{3x+5}$.

$$5x+3 = 3x+5 \Rightarrow x = 1.$$

e.g. Solve $2^{5x+3} = 1024$

$1024 = 2^{10}$

$\Rightarrow 5x+3 = 10 \Rightarrow x = \frac{7}{5}$.

e.g. Solve $4^{5x+3} = 2^{3x+5}$

$4^{5x+3} = (2^2)^{5x+3} = 2^{2\cdot(5x+3)} = 2^{10x+6}$.

$\Rightarrow 10x+6 = 3x+5 \Rightarrow x = -\frac{1}{7}$.

e.g. $\left(\frac{3}{4}\right)^x = \frac{27}{64}$.

$27 = 3^3$, $64 = 4^3 \Rightarrow \frac{27}{64} = \frac{3^3}{4^3} = \left(\frac{3}{4}\right)^3$

$\Rightarrow x = 3$.

Magic number e

Math 120
note.

- Euler's number $\approx 2.718 \ldots$
- irrational
- $(1+1)^1, (1+\frac{1}{2})^2, (1+\frac{1}{3})^3, \ldots (1+\frac{1}{100})^{100} \ldots \longrightarrow e.$
- $e^0 = 1, e^1 = 2.718 \ldots, e^2 = 7.389 \ldots$

Compound interest

pricipal : P (original money)

ending balance : A (money at the end)

APY : $r\%$ (annual percentage yield).

For t years,

annually ~~interest~~ : $A = P \cdot (1+r)^t$

quaterly : $A = P \cdot (1+\frac{r}{4})^{4t}$

monthly : $A = P \cdot (1+\frac{r}{12})^{12t}$

n times : $A = P \cdot (1+\frac{r}{n})^{nt}$

$\vdots$

$\downarrow$

The best $A = P \cdot e^{rt}$

called <u>continuous compounding</u>

Fact : $(1+\frac{r}{n})^n \sim e^r$ when $n \to \infty$.

4.2 4.3 Log functions.

Math 120
note

Motivation: Find x such that $2^x = 8$.

Or Find x such that $1.1^x = 2$.

[That is, with 10% compounding profit,
how many years required to double the money?]

Def: $\log_{\underset{\text{base}}{2}} \underset{\text{value}}{y} = \overset{\text{power}}{x}$ if and only if $\underset{\text{base}}{2}^{\overset{\text{power}}{x}} = \underset{\text{value}}{y}$.

Fixed a base,
(e.g. base = 2)

$$\begin{tikzcd} \underset{\substack{\cancel{\text{value}}\\ \text{power}}}{x} \arrow[r, bend left, "2^x"] & \underset{\text{value}}{y} \arrow[l, bend left, "\log_2 y"] \end{tikzcd}$$

That is $\underline{\log_a(y)}$ is the inverse function of $\underline{a^x}$.

- inverse rule:

$$a^{\log_a x} = \log_a(a^x) = x.$$

when $a = e$, Euler's number 2.718..., $\log_e x = \ln x$.

$$e^{\ln x} = \ln(e^x) = x.$$

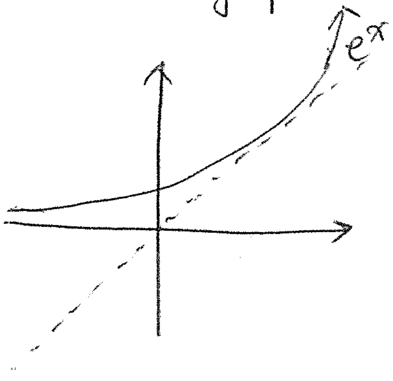
e.g. $\log_2 8 = 3$, $\log_{1.1} 2 = 7.2725...$ (about 7 years to double the money)

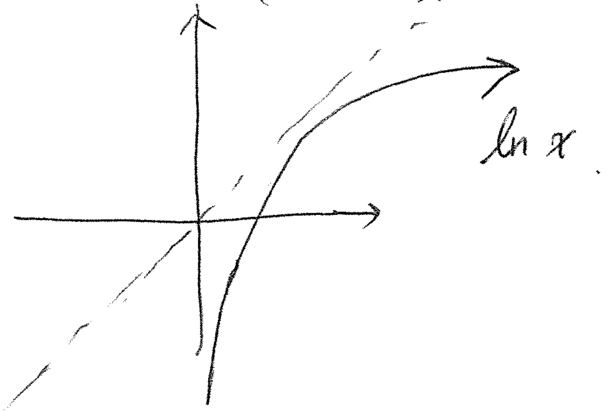
$e^0 = 1 \Leftrightarrow \ln 1 = 0$

$e^1 = e \Leftrightarrow \ln e = 1$

Math 120 note

Recall: inverse function (the graph) is the reflection along $y=x$ (45° line)





~~range: $\mathbb{R}$~~

~~dom~~ domain: $\mathbb{R}$

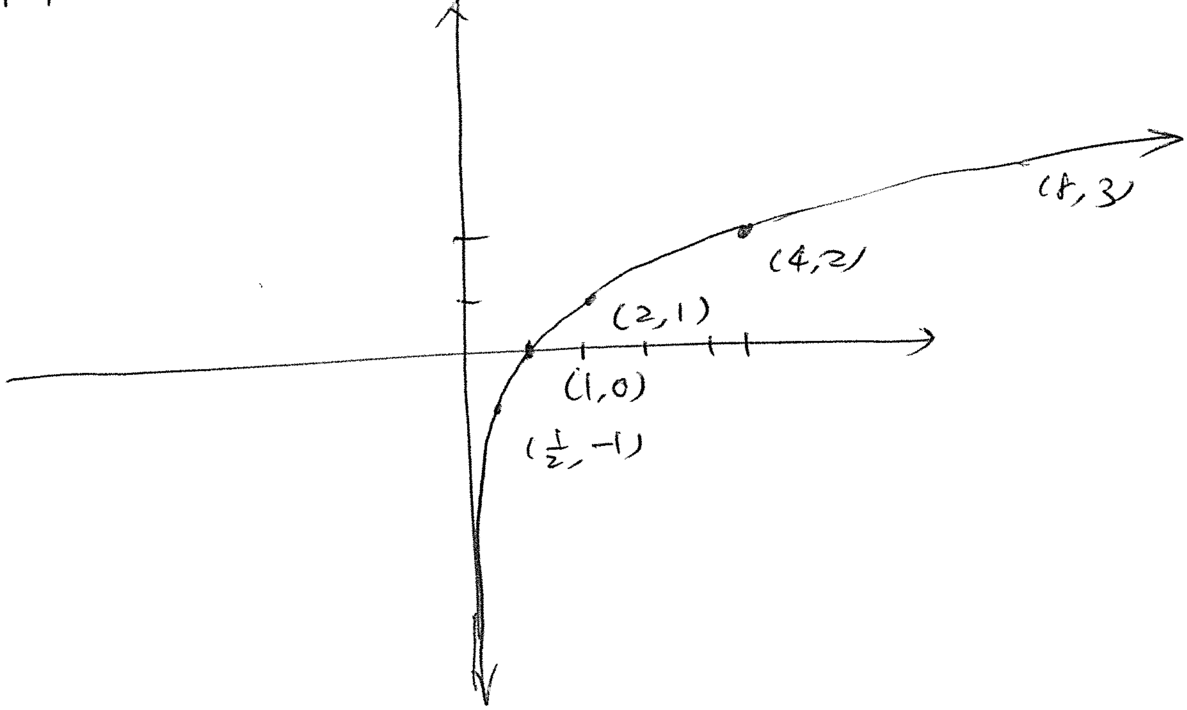
range: $(0,\infty)$

domain: $(0,\infty)$

range: $\mathbb{R}$.

e.g. $f(x) = \log_2(x)$

~~log~~ x	...	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	---	2^y
$\log_2(x)$	...	-3	-2	-1	0	1	2	3	---	y



never passes y-axis

exp rules

$$2^x \cdot 2^y = \overbrace{\underbrace{(2\cdot \cdots \cdot 2)}_{x \text{ times}} \cdot \underbrace{(2 \cdot \cdots \cdot 2)}_{y \text{ times}}}^{x+y \text{ times}} = 2^{x+y}$$

$$(2^x)^y = \underbrace{\overbrace{\underbrace{(2\cdots 2)}_{x \text{ times}} \cdot \underbrace{(2\cdots 2)}_{x \text{ times}} \cdot \cdots \cdot \underbrace{(2\cdots 2)}_{x \text{ times}}}^{xy \text{ times}}}_{y \text{ times.}} = 2^{xy}.$$

log rules

$\log_2(A\cdot B) = \log_2(A) + \log_2(B)$

$\log_2(\frac{A}{B}) = \log_2(A) - \log_2(B)$ (e.g $\log_2(\frac{1}{2}) = -\log_2 2 = -1.$)

$\log_2(A^r) = r\cdot\log_2(A)$ (e.g $\ln(e^{100}) = 100\cdot\ln(e) = 100$).

!!! Always write the base, except for ln.

base change formula

$$\log_a b = \frac{\ln b}{\ln a} = \frac{\log_c b}{\log_c a} \text{ for any } c>0.$$

eg. $\log_{1.1} 2 = \frac{\ln 2}{\ln 1.1} = \frac{0.6931\ldots}{0.0953\ldots} = 7.2725\ldots$

[Calculator usually only has ln and $\log = \log_{10}$]

4.4 Equations.

Math 120 note

Key: apply exp to cancel log

or apply log to cancel exp.

e.g. $3^{x+1} = 3^{2x-3}$

$x+1 = 2x-3$ $\qquad [\log_3(\dots)]$

$x=4$.

e.g. $\log_5 x+1 = \log_5 2x-3$

$x+1 = 2x-3$ $\qquad [5^{(\dots)}]$

$x = 4$.

e.g. ~~Use calculator~~

$\log_2(x-3) = 100$

$x-3 = 2^{100}$ $\qquad [2^{(\dots)}]$

$x = 2^{100}+3$

e.g. $1.01^t = 2$

The same

$t = \log_{1.01} 2$ $\qquad [\log_{1.01}(\dots)]$

Or apply ln on both sides.

$\ln(1.01^t) = \ln(2)$

$t \cdot \ln(1.01) = \ln(2) \Rightarrow t = \dfrac{\ln(2)}{\ln(1.01)}$

e.g. $3^{2x-1} = 5^{x}$

$\ln(3^{2x-1}) = \ln(5^{x})$ 取 $[\ln(\cdots)]$

$(2x-1)\cdot\ln(3) = x\cdot\ln(5)$

$(2\ln 3)\cdot x - \ln 3 = (\ln 5)\, x.$

$(2\ln 3 - \ln 5)\, x = \ln 3$

$$x = \frac{\ln 3}{2\ln 3 - \ln 5}$$

e.g. $\ln x^{2} + \ln x^{3} = 5$

$2\cdot\ln x + 3\cdot\ln x = 5$

$5\ln x = 5$

$\ln x = 1 \Rightarrow x = e.$

Application : Population model.

~~growing const~~ growth constant

$$A = A_{0}\, e^{rt}$$

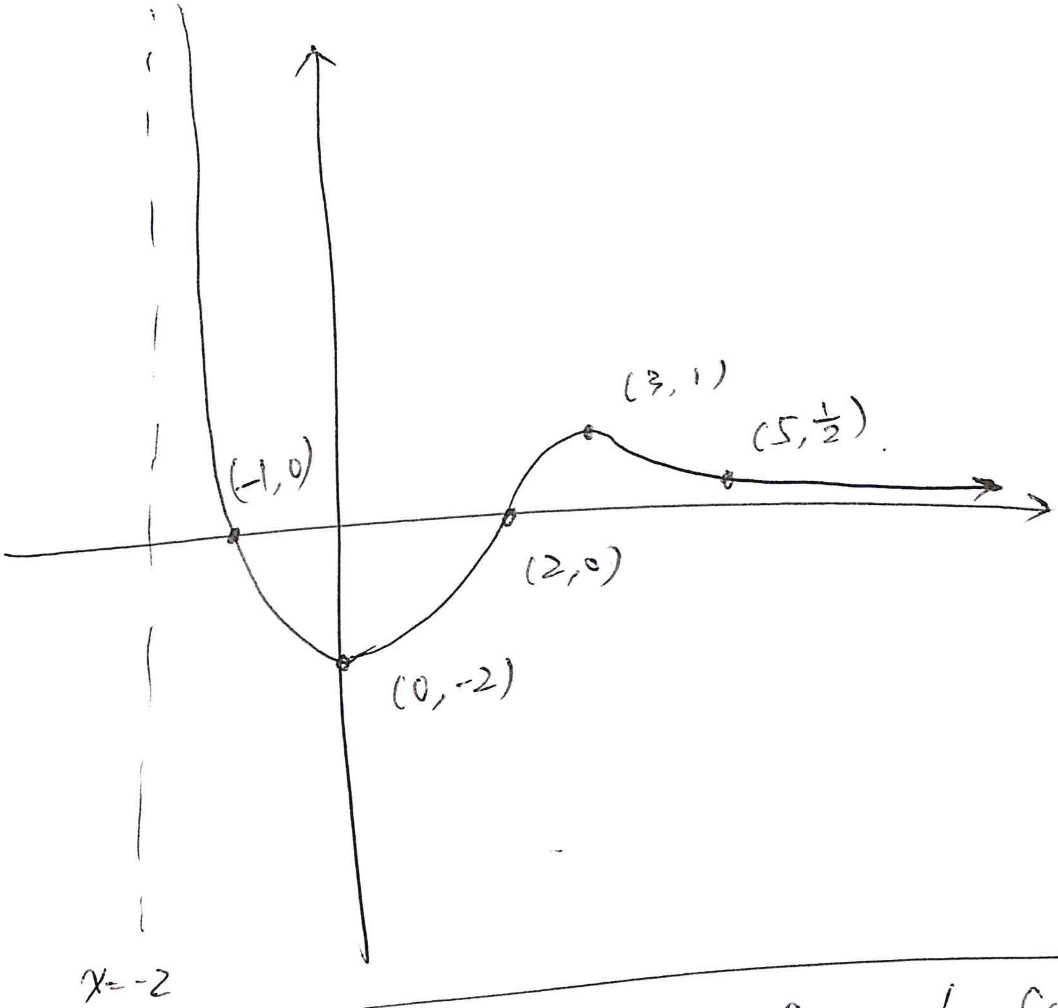
$rt \leftarrow$ years.

ending population

initial population

Functions

algebraic	graphic
$x \longmapsto f(x)$	points $(x, f(x))$
domain: all possible x	width
range: all possible y.	height.
being a function: x determines $f(x)$	vertical test
one-to-one: $f(x)$ determines x	horizontal test.
y-intercept	$(0, f(0))$
x-intercept(s)	$(b, 0)$ with any $f(b)=0$.
not defined	$x=a$ does not touch the graph
$f(x)>0$	above x-axis
$f(x)<0$	below x-axis
$f(x)=0$	~~points~~ values of x where the graph touches the x-axis (a.k.a. x-intercep
increasing	↗
decreasing	↘
$f^{-1}(x)$	reflection of $f(x)$ along $y=x$
$f(x-a)+b$	translation of $f(x)$ to (a,b)
$b \cdot f\left(\frac{x}{a}\right)$	⟷ by a and ↕ by b.



$f(-1) = 0$, ~~f(0)~~ $f(0) = -2$, $f(2) = 0$, $f(5) = \frac{1}{2}$, $f(3) = 1$.

function? Yes

domain : $(-2, \infty)$

One-to-one? No. ⇒ no inverse.

range : $(-2, \infty)$

y-intercept : $(0, -2)$

x-intercept(s) : $(-1, 0)$, $(2, 0)$ opoints

not defined : any x with $x \leq -2$.

$f(x) > 0$: $(-2, -1)$, $(2, \infty)$ $-2 \leq x < -1$ or $2 < x < \infty$

or ∪

$f(x) < 0$: $(-1, 2)$

$f(x) = 0$: -1, 2

increasing : $(0, 3)$

decreasing : ~~-1, 0~~ $(-2, 0)$, $(3, \infty)$