Math555 Midterm 2

Note: Use other papers to answer the problems. Remember to write down your **name** and your **student ID #**.

1. [4pt] Let $\phi(n)$ be the Euler's totient function. That is, $\phi(n)$ is the number of integers k with $\gcd(k, n) = 1$ and $1 \leqslant k \leqslant n$. Consider $n = 16$ and the set $[n] = \{1, \ldots, 16\}$. Let
$$A_d = \{k \in [n] : \gcd(k, n) = d\}.$$
For each $d \mid n$, find A_d and verify $|A_d| = \phi(n/d)$.

2. [4pt] Let $\mu(n)$ be the Möbius function. Prove that
$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

3. [4pt] Suppose x_n is an integer for every $n \geqslant 1$ and
$$y_n = \sum_{d|n} x_d.$$
If $y_n = n^2$ for every $n \geqslant 1$. Use Möbius inversion to find x_{24}.

4. [4pt] Solve the recurrence relation below.
$$\begin{cases} a_n + 0a_{n-1} - 3a_{n-2} - 2a_{n-3} = 0. \\ a_0 = 4, a_1 = -3, a_2 = 11. \end{cases}$$

5. [4pt] Find the closed form of the generating function $\sum_{k \geqslant 0} k^2 x^k$.