Math555 Homework 6

Note: To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Define a function $\nu(n)$ by the following recurrence relation.

$$\nu(n) = \begin{cases} 1 & \text{if } n = 1; \\ -\sum_{\substack{d|n \\ d\neq n}} \nu(d) & \text{otherwise.} \end{cases}$$

Show that $\nu(n) = \mu(n)$ for all $n \geqslant 1$.

Solution. Prove by induction. For the base step, $\mu(1) = 1 = \nu(1)$. Suppose $\mu(k) = \nu(k)$ for all values $k < n$. We show that $\mu(n) = \nu(n)$.

The Möbius function has the property $\sum_{d|n} \mu(d) = 0$ for any $n \geqslant 2$. Therefore, $\mu(n) = -\sum_{\substack{d|n \\ d\neq n}} \mu(d)$. Now by induction hypothesis, $\mu(d) = \nu(d)$ for any $d \mid n$. Thus,

$$\nu(n) = -\sum_{\substack{d|n \\ d\neq n}} \nu(d) = -\sum_{\substack{d|n \\ d\neq n}} \mu(d) = \mu(n).$$

By induction, $\mu(n) = \nu(n)$ for all $n \geqslant 1$.

2. Use Sage to write a function for the Euler's totient function $\phi(n)$ and another function for the Möbius function $\mu(n)$. See the file `SageProject1_blank.sagews` in your CoCalc folder.

   Solution. The sample solutions are posted on the course website.