Math555 Homework 4

Note: To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Let $S(n, k)$ denote the Stirling number of the second kind that counts the number of partitions of $[n]$ into k parts and let $B(n)$ be the Bell number that counts the number of total partitions of $[n]$, where $B(0) = 1$. That is,

$$B(n) = \sum_{k=0}^{n} S(n, k).$$

Prove each of the following identities.

(a) $B(n+1) = \sum_{i=0}^{n} \binom{n}{n-i} B(i) = \sum_{i=0}^{n} \binom{n}{i} B(i).$

(b) $\sum_{n \geqslant 0} B(n) \frac{x^n}{n!} = \exp(e^x - 1).$

Solution.

(a) Consider $B(n+1)$ as the number of partitions of $[n+1]$. In each partition, the element $n+1$ must be contained in a set $I \cup \{n+1\}$ for some $I \subseteq [n]$. So

$$\begin{aligned} B(n+1) &= \sum_{I \subseteq [n]} (\text{\# partitions of } [n] \setminus I) \\ &= \sum_{I \subseteq [n]} B(n - |I|) = \sum_{i=0}^{n} \binom{n}{i} B(n-i) \\ &= \sum_{i=0}^{n} \binom{n}{n-i} B(i) = \sum_{i=0}^{n} \binom{n}{i} B(i). \end{aligned}$$

(b) By the lecture note, we know

$$F_k(x) = \sum_{n \geqslant k} S(n, k) \frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k.$$

Therefore, by the definition of $B(n)$,

$$\sum_{n \geqslant 0} B(n) \frac{x^n}{n!} = \sum_{k \geqslant 0} F_k(x) = \exp(e^x - 1).$$

Notice that $F_0(x) = 1$.

2. For any positive integer n, let D_n denote the number of derangements of $[n]$. Define $D_0 = 1$.

   (a) Prove that, for $n \geqslant 1$, D_n is the closest integer to $\frac{n!}{e}$.

   (b) Prove that, for $n \geqslant 2$, $D_n = (n-1)(D_{n-1} + D_{n-2})$.

   (c) Prove that, for $n \geqslant 1$, $D_n = nD_{n-1} + (-1)^n$.

   You may prove (b) algebraically, but it is probably easier to prove it combinatorially. You may prove (c) combinatorially, but it is probably easier to prove it algebraically.

   Solution.

   (a) By the lecture note, we know

   $$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

   Therefore,

   $$\begin{aligned} \left| D_n - \frac{n!}{e} \right| &= \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k n!}{k!} \right| \\ &= \frac{n!}{(n+1)!} - \left(\frac{n!}{(n+2)!} - \frac{n!}{(n+3)!} \right) - \cdots \quad < \frac{1}{n+1} \end{aligned}$$

   since it is an alternating sequence. Thus, this error term is less than $\frac{1}{2}$ when $n \geqslant 1$.

   (b) Consider D_n as the number of derangement on $[n]$. For any derangement π, $\pi(n) \in [n-1]$ (so $n-1$ choices). Suppose $\pi(n) = j$. Then there are two cases. First, if $\pi(j) = n$, then the permutation of π induced on $[n] \setminus \{n, j\}$ is a derangement on $[n] \setminus \{n, j\}$ (so D_{n-2} of them). The second case will be $\pi(j) \neq n$. Now π induces a map from $[n-1]$ to $[n] \setminus \{j\}$; this map can be viewed as a derangement by thinking the element n in the range as j (so D_{n-1} of them). Overall, there are $(n-1)(D_{n-1} + D_{n-2})$ derangements.

   (c) We may compute that

   $$\begin{aligned} nD_{n-1} + (-1)^n &= n(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + (-1)^n \\ &= n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + \frac{(-1)^n n!}{n!} \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = D_n. \end{aligned}$$