Math555 Homework 4

Note: To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Let $S(n,k)$ denote the Stirling number of the second kind that counts the number of partitions of $[n]$ into k parts and let $B(n)$ be the Bell number that counts the number of total partitions of $[n]$, where $B(0) = 1$. That is,

$$B(n) = \sum_{k=0}^{n} S(n,k).$$

   Prove each of the following identities.

   (a) $B(n+1) = \sum_{i=0}^{n} \binom{n}{n-i} B(i) = \sum_{i=0}^{n} \binom{n}{i} B(i).$

   (b) $\sum_{n \geqslant 0} B(n) \frac{x^n}{n!} = \exp\left(e^x - 1\right).$

2. For any positive integer n, let D_n denote the number of derangements of $[n]$. Define $D_0 = 1$.

   (a) Prove that, for $n \geqslant 1$, D_n is the closest integer to $\frac{n!}{e}$.

   (b) Prove that, for $n \geqslant 2$, $D_n = (n-1)(D_{n-1} + D_{n-2})$.

   (c) Prove that, for $n \geqslant 1$, $D_n = nD_{n-1} + (-1)^n$.

   You may prove (b) algebraically, but it is probably easier to prove it combinatorially. You may prove (c) combinatorially, but it is probably easier to prove it algebraically.