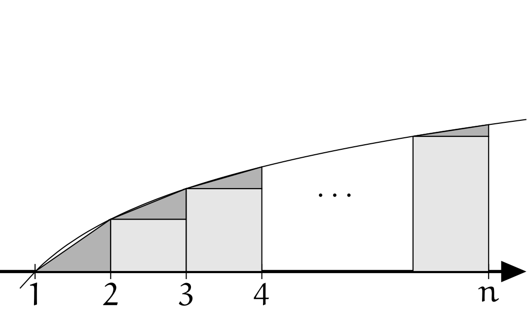
Math555 Homework 2

Note: You may turn in your homework through paper work (first three weeks only) or through CoCalc. To submit the k-th homework, simply put your files in the folder `HWk` on CoCalc, and it will be collected on the due day.

1. Show that
$$\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \frac{n^k}{k!} \leqslant \left(\frac{en}{k}\right)^k$$
for any integers n and k with $1 \leqslant k \leqslant n$.

   Solution. For the first inequality, since $\frac{n-i}{k-i} \geqslant \frac{n}{k}$ for $i < k \leqslant n$,
$$\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-k+1}{1} \geqslant \left(\frac{n}{k}\right)^k.$$
For the second inequality,
$$\binom{n}{k} = \frac{(n)_k}{k!} \leqslant \frac{n^k}{k!}.$$
For the third inequality, observe that
$$e^k = \sum_{i=0}^{\infty} \frac{k^i}{i!} \geqslant \frac{k^k}{k!}.$$
Then substitute the inequality $k! \leqslant \frac{k^k}{e^k}$ to $\frac{n^k}{k!}$ and get the desired inequality. Note that this is a weaker result than the Stirling's formula, so try not to use Stirling's formula to prove it.

2. Show that
$$n! \leqslant \frac{e\sqrt{n} \cdot n^n}{e^n}$$
for any integer $n \geqslant 1$, where e is Euler's number.
Hint: First explain that $\ln(n-1)! + \frac{1}{2}\ln n \leqslant \int_1^n \ln x \, dx$. Then remember the fact that $n! = n \cdot (n-1)!$.



Solution. Following the graph above, the total area of the white rectangles and the gray triangles is

$$\begin{aligned}&\ln(n-1)! + \frac{1}{2}(\ln 2 - \ln 1) + \frac{1}{2}(\ln 3 - \ln 2)) + \cdots + \frac{1}{2}(\ln n - \ln(n-1))\\ =& \ln(n-1)! + \frac{1}{2}\ln n.\end{aligned}$$

Thus, we have
$$\begin{aligned}\ln(n-1)! + \frac{1}{2}\ln n &\leqslant \int_1^n \ln x \, dx\\ &= [x\ln x - x]\Big|_1^n\\ &= [n\ln n - 0] - [n-1]\\ &= \ln\frac{en^n}{e^n}.\end{aligned}$$

This means
$$(n-1)! \leqslant \frac{en^n}{\sqrt{n}e^n}.$$
Multiplying the both sides by n gives the desired result.