Math555 Final

8 questions, 40 total points
Note: Use other papers to answer the problems. Remember to write down your **name** and your **student ID #**.

1. [5pt] Let N and X be two sets with $|N| = 2$ and $|X| = 3$. Fill in the following table by the number of functions $f : N \to X$ with the given conditions.

N	X	Any f	injective f	surjective f
dist	dist	(i)	(ii)	(iii)
indist	dist	(iv)	(v)	(vi)
dist	indist	(vii)	(viii)	(ix)
indist	indist	(x)	(xi)	(xii)

Solution. Use the formulas given in the lecture notes.

N	X	Any f	injective f	surjective f
dist	dist	$3^2 = 9$	$(3)_2 = 6$	0
indist	dist	$\left(\!\binom{3}{2}\!\right) = 6$	$\binom{3}{2} = 3$	0
dist	indist	$S(2,0) + S(2,1) + S(2,2) + S(2,3) = 2$	1	0
indist	indist	$p_0(2) + p_1(2) + p_2(2) + p_3(2) = 2$	1	0

2. [5pt] Solve the recurrence relation below.

$$\begin{cases} a_n + 0a_{n-1} - 3a_{n-2} - 2a_{n-3} = 0. \\ a_0 = 4, a_1 = 3, a_2 = 17. \end{cases}$$

Solution. The characteristic polynomial is

$$p(x) = x^3 - 3x - 2 = (x+1)^2(x-2)$$

with the roots $-1, -1, 2$. Thus, the formula for a_n is

$$a_n = A \cdot (-1)^n + B \cdot n(-1)^n + C \cdot 2^n.$$

Substituting this equality with $n = 0, 1, 2$, we get the following equations.

$$\begin{cases} A + C = 4 \\ (-1)A + (-1)B + 2C = 3 \\ A + 2B + 4C = 17 \end{cases}$$

It follows that $A = 1$, $B = 2$, and $C = 3$, so

$$a_n = (-1)^n + 2n(-1)^n + 3 \cdot 2^n.$$

3. [5pt] Find the first ten terms of the reciprocal of $f(x) = 1 - x + x^2$.

Solution. Let $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$. Suppose $f(x)g(x) = 1$. Direct computation gives the following.

$$
\begin{aligned}
1 &= 1b_0 \implies b_0 = 1 \\
0 &= 1b_1 - 1b_0 \implies b_1 = 1 \\
0 &= 1b_2 - 1b_1 + 1b_0 \implies b_2 = 0 \\
0 &= 1b_3 - 1b_2 + 1b_1 \implies b_3 = -1 \\
0 &= 1b_4 - 1b_3 + 1b_2 \implies b_4 = -1 \\
0 &= 1b_5 - 1b_4 + 1b_3 \implies b_5 = 0 \\
0 &= 1b_6 - 1b_5 + 1b_4 \implies b_6 = 1 \\
0 &= 1b_7 - 1b_6 + 1b_5 \implies b_7 = 1 \\
0 &= 1b_8 - 1b_7 + 1b_6 \implies b_8 = 0 \\
0 &= 1b_9 - 1b_8 + 1b_7 \implies b_9 = -1
\end{aligned}
$$

Thus,

$$g(x) = 1 + x + 0x^2 - x^3 - x^4 + 0x^5 + x^6 + x^7 + 0x^8 - x^9 + \cdots .$$

4. [5pt] Let $f(x) = (1+x)^{-1}$. Find a_k such that $f''(x) = \sum_{k \geqslant 0} a_k x^k$. Here $f''(x)$ is the second (formal) derivative of $f(x)$.

Solution. Compute $f''(x) = (-1)(-2)(1+x)^{-3} = 2(1+x)^{-3}$. So $a_k = 2\binom{-3}{k}$.

5. [5pt] Compute

$$A = \sum_{n \geqslant 0} \frac{n^2}{n!}, B = \sum_{n \geqslant 0} \frac{n}{n!}, \text{ and } C = \sum_{n \geqslant 0} \frac{1}{n!}.$$

Then find the value of

$$\sum_{n \geqslant 0} \frac{2n^2 + 4n - 3}{n!}.$$

Solution. Let $f_0(x) = e^x = \sum_{n \geqslant 0} \frac{x^n}{n!}$. Then $C = f_0(1) = e$. Next, compute

$$f_1(x) = (xD)f_0(x) = xe^x = \sum_{n \geqslant 0} \frac{nx^n}{n!}.$$

Therefore, $B = f_1(1) = e$. Again, compute

$$f_2(x) = (xD)f_1(x) = e^x + xe^x = \sum_{n \geqslant 0} \frac{n^2 x^n}{n!}.$$

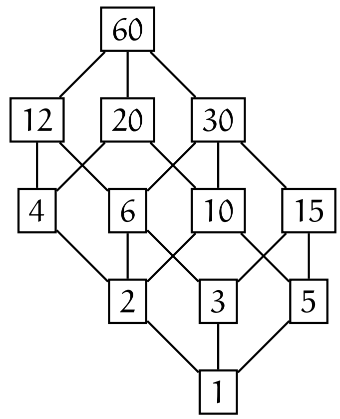
Thus, $A = f_2(1) = 2e$.

Finally, the desired value is

$$2A + 4B - 3C = 5e.$$

6. [5pt] Draw the Hasse diagram for the poset D_{60}.

Solution.



7. [5pt] Prove that the poset D_{30} and the poset D_{105} are isomorphic. That is, find a bijection between the factors of 30 and the factors of 105 that preserve the relation.

Solution. Note that

$$30 = 2 \cdot 3 \cdot 5 \text{ and } 105 = 3 \cdot 5 \cdot 7.$$

The factors of 30 are

$$S = \{2^a \cdot 3^b \cdot 5^c : a, b, c \in \{0, 1\}\}$$

and the factors of 105 are

$$T = \{3^a \cdot 5^b \cdot 7^c : a, b, c \in \{0, 1\}\}.$$

Define the map $f : S \to T$ by $f(2^a \cdot 3^b \cdot 5^c) = 3^a \cdot 5^b \cdot 7^c$. One may check that f is an isomorphism between D_{30} and D_{105}.

8. [5pt] Consider the poset D_8. Find the matrix forms of the zeta function and the Möbius function on D_8, using $\{1, 2, 4, 8\}$ as the index of the matrix.

Solution. For the zeta function,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the Möbius function,

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$