

Sample Solutions for Sample Questions 8.

1.

$$A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 2 & 0 & 3 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 12 & 5 \\ 0 & 0 & 11 & 6 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & 6 \\ 0 & 0 & 11 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 11 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row space :

basis

$$\langle (1, 0, -4, -1), (0, 1, 1, -1), (0, 0, 11, 6) \rangle$$

dimension = 3 \leftarrow row rank.

Column space :

basis $\left\langle \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ -4 \end{pmatrix} \right\rangle$

dimension = 3 \leftarrow column rank.

Null space :

basis $\left\langle \begin{pmatrix} -13/11 \\ 17/11 \\ -6/11 \\ 1 \end{pmatrix} \right\rangle$

(by setting $w = 1$ and solve $A \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 0$.)

dimension = 1 \leftarrow nullity

2.

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -5 & 3 & -4 \\ 0 & 1 & 1 & 4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & -5 & 3 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 8 & 16 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & -1 & 2 \\ & 1 & 1 & 4 \\ & & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -4 & -10 \\ & 1 & 1 & 4 \\ & & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 \\ & 1 & 0 & 2 \\ & & 1 & 2 \end{pmatrix}$$

Row space:

basis:

$$\left\langle (1, 0, 0, -2), (0, 1, 0, 2), (0, 0, 1, 2) \right\rangle$$

dimension = 3 ← rank

Column space:

$$\text{basis: } \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

dimension = 3 ← rank.

Null space:

$$\text{basis: } \left\langle \begin{pmatrix} 2 \\ -2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$$

dimension = 1 ← nullity.

3.

Since $a \neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ 0 & d - b \cdot \frac{c}{a} \end{pmatrix}.$$

To have $\text{rank} = 1$,

$$d - b \cdot \frac{c}{a} = 0 \Rightarrow d = b \cdot \frac{c}{a}.$$

4. Claim: consistent $\Leftrightarrow \text{rank}(A) = \text{rank}([A|b])$.

Proof 1:

Suppose R is the reduced echelon form of A .

Then $[R|r]$ is the reduced echelon form of $[A|b]$

for some column vector r .

consistent means $\#$ of nonzero rows in R
 $= \#$ of nonzero rows in $[R|r]$

Proof 2:

$Ax = b$ is consistent $\Leftrightarrow b \in \text{Columnspace}(A)$

$$\Leftrightarrow \text{Colspace}(A) = \text{Colspace}([A|b]).$$

$$\Leftrightarrow \dim \text{Colspace}(A) = \dim \text{Colspace}([A|b])$$

$$\Leftrightarrow \text{rank}(A) = \text{rank}([A|b])$$

5.

(a).

$Ax = b$ is consistent for all $b \in \mathbb{R}^m$

$\Leftrightarrow b \in \text{Colspace}(A)$ for all $b \in \mathbb{R}^m$

$\Leftrightarrow \text{Colspace}(A) = \mathbb{R}^m$

$\Leftrightarrow \dim \text{Colspace}(A) = m$

$\Leftrightarrow \text{rank}(A) = m \leftarrow \# \text{ of rows}$

(b).

$Ax = b$ has a unique solution for all $b \in \mathbb{R}^m$
making $Ax = b$ consistent.

$\Leftrightarrow Ax = b$ has a unique solution for all $b \in \text{Colspace}(A)$.

\Leftrightarrow every $b \in \text{Colspace}(A)$ has a unique representation
with respect to columns in A .

\Leftrightarrow columns in A form a linearly indep. set.

$\Leftrightarrow \dim \text{Colspace}(A) = n$.

$\Leftrightarrow \text{rank}(A) = n \leftarrow \# \text{ of columns}$

6. ① Let S_A and S_B be bases of $\text{Colspace}(A)$ and $\text{Colspace}(B)$, respectively.

$$\text{Let } A = \begin{bmatrix} | & | & \dots & | \\ a_1 & a_2 & \dots & a_n \\ | & | & \dots & | \end{bmatrix}, \quad B = \begin{bmatrix} | & | & \dots & | \\ b_1 & b_2 & \dots & b_n \\ | & | & \dots & | \end{bmatrix}.$$

$$\text{Then } A+B = \begin{bmatrix} | & | & \dots & | \\ a_1+b_1 & a_2+b_2 & \dots & a_n+b_n \\ | & | & \dots & | \end{bmatrix}.$$

$$\text{So } a_i+b_i = 1 \cdot a_i + 1 \cdot b_i \in \text{span}(S_A \cup S_B).$$

Therefore, $\text{Colspace}(A+B) \subseteq \text{span}(S_A \cup S_B)$.

$$\begin{aligned} \Rightarrow \text{rank}(A+B) &= \dim \text{Colspace}(A+B) \\ &\leq \dim \text{span}(S_A \cup S_B) \\ &\leq |S_A \cup S_B| = \text{rank}(A) + \text{rank}(B). \end{aligned}$$

②

Each column of AB is a linear combination of columns

$$\text{of } A \Rightarrow \text{Colspace}(AB) \subseteq \text{Colspace}(A)$$

$$\Rightarrow \text{rank}(AB) \leq \text{rank}(A).$$

Similarly, each ~~column~~ row of AB is a linearly combination

$$\text{of rows in } B \Rightarrow \text{Rowspace}(AB) \subseteq \text{Rowspace}(B)$$

$$\Rightarrow \text{rank}(AB) \leq \text{rank}(B).$$

7. Let $B = \langle \vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_n \rangle$ be a basis of V .

Let $D = \langle \vec{\delta}_1, \vec{\delta}_2, \dots, \vec{\delta}_m \rangle$ be a linearly indep. set.

Since B is a basis, B is indep. and $\text{span}(B) = V$.

① $\vec{\delta}_1 \in \text{span}(B) \Rightarrow$ there is $\hat{\beta}_1 \in B$ such that

$$B_1 = \langle \cancel{B} \cup \{ \vec{\delta}_1 \} \setminus \{ \hat{\beta}_1 \} \rangle$$

$B_1 = B \cup \{ \vec{\delta}_1 \} \setminus \{ \hat{\beta}_1 \}$ is a basis,

by Exchange Lemma.

② $\vec{\delta}_2 \in \text{span}(B_1) \Rightarrow$ there is $\hat{\beta}_2 \in B$ such that
 ($\because B_1$ is a basis)

$B_2 = B_1 \cup \{ \vec{\delta}_2 \} \setminus \{ \hat{\beta}_2 \}$ is a basis.

\vdots

③ $\vec{\delta}_k \in \text{span}(B_{k-1}) \Rightarrow \vec{\delta}_k = c_1 \vec{\delta}_1 + \dots + c_{k-1} \vec{\delta}_{k-1} + \overbrace{\text{linear combination}}^{\vec{b}}$

($\because B_{k-1}$ is a basis) where \vec{b} is a linear combination of the $\vec{\beta}_i$'s in B_{k-1} .

Since $\{ \vec{\delta}_1, \dots, \vec{\delta}_k \}$ is indep., $\vec{b} \neq 0$.

\Rightarrow there is $\hat{\beta}_k \in B$ such that

$B_k = B_{k-1} \cup \{ \vec{\delta}_k \} \setminus \{ \hat{\beta}_k \}$ is a basis.

\vdots

④ B_m is a basis ~~but~~ of the form

$$\langle \underbrace{\vec{\delta}_1, \dots, \vec{\delta}_m}_D, \underbrace{n-m \text{ vectors in } B}_{B'} \rangle$$

$$= D \cup B'$$

Thus, this completes the proof.