Sample Questions 7

1. For each of the following vector spaces, find a basis and the dimension.
   (a) the space of all 2×2 matrices
   (b) the space of all polynomials of degree at most 3
   (c) $\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : c - 2b = 0 \right\}$
   (d) $\operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$
   (e) $\operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \right\}$
   (f) the solution set of
   $$\begin{cases} x_1 - 4x_2 + 3x_3 - x_4 = 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 = 0 \end{cases}$$

2. Suppose $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ is a basis for a vector space. Show that $\langle c_1\mathbf{x}_1, c_3\mathbf{x}_2, c_3\mathbf{x}_3 \rangle$ is a basis when $c_1, c_2, c_3 \neq 0$. Also, show that $\langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \rangle$ is a basis where $\mathbf{y}_i = \mathbf{x}_1 + \mathbf{x}_i$ for $i = 1, 2, 3$.

3. A square matrix is *symmetric* if its i, j-entry equals its j, i-entry for every pair i, j. Find a basis for the space of all symmetric 3×3 matrices.

4. Let $B = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\rangle$ be a basis in $\mathbb{R}^2$. Treating the vectors in B as the x-axis and the y-axis, respectively, sketch the coordinate systems (the grid lines) corresponding to B. Then find $\operatorname{Rep}_B(\begin{bmatrix} 2 \\ 4 \end{bmatrix})$ and $\operatorname{Rep}_B(\begin{bmatrix} -4 \\ 1 \end{bmatrix})$.

5. Find a vector $\mathbf{v}$ such that B is a basis of V.
   (a) $B = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v} \right\rangle$, $V = \mathbb{R}^2$
   (b) $B = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} \right\rangle$, $V = \mathbb{R}^3$
   (c) $B = \langle x, 1 + x^2, \mathbf{v} \rangle$, V is the space of all polynomials of degree at most 2

6. Given a homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$, the algorithm for finding the general solution (Page 11, lecture notes 2) outputs k vectors $S = \langle \beta_1, \ldots, \beta_k \rangle$ such that $\operatorname{span}(S)$ is the solution set and k is the number of free variables. Show that S is a linearly independent set, so that S is a basis for the solution set. [Hint: If x_i is a free variable, what is the value of x_i on each of the vectors in S?]