

Sample Solutions for Sample Questions 5.

1.

(a) Yes. $S = \text{span}\left(\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}\right)$.

[Since S is the span of some vectors,
 S is a vector space.]

(b) No. S does not contain a zero vector.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin S.$$

(c) Yes. $S = \text{span}\left(\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}\right)$.

To solve $a+b=0$,

$$\text{let } b=s \rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -s \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

2. (a) Intuitively, $v = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \text{span}(S)$.

In general, solve

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

$$\Rightarrow a=2, b=1.$$

(b)

Solve

$$a \cdot (0 + 0x + 1x^2 + 0x^3)$$

$$b \cdot (0 + 2x + 1x^2 + 0x^3)$$

$$+ c \cdot (0 + 4x + 0x^2 + 1x^3)$$

$$0 + 1x + 0x^2 - 1x^3$$

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \begin{matrix} a = -1 \\ b = 1 \\ c = -1 \end{matrix}$$

S_0

$$x - x^3 = -1 \cdot (x^2)$$

$$+ 1 \cdot (2x + x^2)$$

$$- 1 \cdot (x + x^3) \in \text{span}(S).$$

(c)

Solve

$$a \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \text{no solution.}$$

$$S_0 \quad v \notin \text{span}(S).$$

3.

(a) Solve $a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

any vector in \mathbb{R}^3 .

augmented matrix

$$\left(\begin{array}{ccc|c} 1 & & & x \\ & 2 & & y \\ & & 3 & z \end{array} \right) \text{ always has solution}$$

regardless of the choice of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Yes, $\text{span}(S) = \mathbb{R}^3$.

(b) Solve $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & x \\ 0 & 1 & 3 & y \\ 1 & 1 & 5 & z \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & x \\ 0 & 1 & 3 & y \\ 0 & 1 & 3 & z-x \end{array} \right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & x \\ 0 & 1 & 3 & y \\ 0 & 0 & 0 & z-x-y \end{array} \right)$$

The linear system has no solution if $z-x-y \neq 0$,

so $\text{span}(S) \neq \mathbb{R}^3$.

[e.g. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin \text{span}(S)$ since $z-x-y = 0-1-0 = -1 \neq 0$.]

(c)

$$\text{Solve } \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & x \\ 0 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 0 & x-y \\ 0 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 0 & x-y \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z - \frac{1}{2}(x-y) \end{array} \right) \text{ always has}$$

Solution regardless of the choice of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\text{So } \text{span}(S) = \mathbb{R}^3.$$

4. Let $E = \{ \text{all even functions} \}$.

$\mathcal{O} = \{ \text{all odd functions} \}$.

① Claim: E is a subspace.

Suppose f, g are even functions and $r \in \mathbb{R}$.

$$\text{Then } \begin{cases} f(-x) = f(x) \\ g(-x) = g(x) \end{cases}$$

$$\text{So } f(-x) + g(-x) = f(x) + g(x) \Leftrightarrow f+g \in E.$$

$$r \cdot f(-x) = f(x) \Leftrightarrow rf \in E.$$

② Claim: \mathcal{O} is a subspace.

$$\text{Suppose } f, g \in \mathcal{O} \text{ and } r \in \mathbb{R} \Rightarrow \begin{cases} f(-x) = -f(x) \\ g(-x) = -g(x) \end{cases}$$

$$\Rightarrow f(-x) + g(-x) = -[f(x) + g(x)] \Rightarrow f+g \in \mathcal{O}.$$

$$\Rightarrow r f(-x) = -r f(x) \Rightarrow rf \in \mathcal{O}.$$

5. Equivalently, we show

$$c \cdot v = 0 \text{ for all } v \in S \iff c \cdot v = 0 \text{ for all } v \in \text{span}(S)$$

" \Rightarrow " Let $u \in \text{span}(S)$.

$$\text{Then } u = a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_k \cdot v_k, \quad a_1, \dots, a_k \in \mathbb{R}, \quad v_1, \dots, v_k \in S.$$

We know $c \cdot v_i = 0$ for all v_i .

$$\begin{aligned} \Rightarrow c \cdot u &= c \cdot (a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_k \cdot v_k) \\ &= a_1 (c \cdot v_1) + a_2 (c \cdot v_2) + \dots + a_k (c \cdot v_k) \\ &= a_1 \cdot 0 + a_2 \cdot 0 + \dots + a_k \cdot 0 = 0. \end{aligned}$$

Since the choice of u is arbitrary,
 $c \cdot u = 0$ for all $u \in \text{span}(S)$.

" \Leftarrow " This is immediate because $S \subseteq \text{span}(S)$.

6.

$Ac=b$ is equivalent to the equation

$$c_1 \begin{pmatrix} | \\ v_1 \\ | \end{pmatrix} + c_2 \begin{pmatrix} | \\ v_2 \\ | \end{pmatrix} + \dots + c_n \begin{pmatrix} | \\ v_n \\ | \end{pmatrix} = \begin{pmatrix} | \\ b \\ | \end{pmatrix}$$

By definition $b \in \text{span}(S)$ if and only if
the solution for c_1, \dots, c_n exists.

7. $Ac=b$ is equivalent to

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0.$$

By definition, S is linearly independent

$\Leftrightarrow c_1 = c_2 = \dots = c_n = 0$ is the unique solution.