Sample Solutions for Sample Questions 3.

1. $$\left(\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & 5 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 2 & -1 & 0 & -1 \\ 0 & 1 & 2 & 5 \end{array}\right)$$

$$\xrightarrow[-2\rho_1+\rho_2]{\cancel{\rho_1}} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & -7 & 2 & -11 \\ 0 & 1 & 2 & 5 \end{array}\right) \xrightarrow{\rho_2 \leftrightarrow \rho_3} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & 5 \\ 0 & -7 & 2 & -11 \end{array}\right)$$

$$\xrightarrow[7\rho_2+\rho_3]{-3\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & -7 & -10 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 16 & 24 \end{array}\right) \xrightarrow{\frac{1}{16}\rho_3} \left(\begin{array}{ccc|c} 1 & 0 & -7 & -10 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\xrightarrow[-2\rho_3+\rho_2]{7\rho_3+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & \cancel{0}1 & \frac{3}{2} \end{array}\right)$$

so $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 \\ 2 \\ 3/2 \end{pmatrix}.$$

General solution $= \left\{ \begin{pmatrix} 1/2 \\ 2 \\ 3/2 \end{pmatrix} \right\}$

a set of the unique solution

2.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 2 & -1 & -1 & 1 \\ 3 & 0 & -2 & 4 \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & -3 & 1 & -5 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

free variable.

Let $z = s$.

$$x = \frac{4}{3} + \frac{2}{3}s$$

$$y = \frac{5}{3} + \frac{1}{3}s$$

$$z = 0 + s$$

$$\text{General solution} = \left\{ \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \\ 0 \end{pmatrix} + s \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} : s \in \mathbb{R} \right\}$$

3.
$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 1 & 1 \\ 3 & 0 & 3 & 2 & 1 \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & -1 & 1 \\ 0 & -3 & -3 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}\rho_2} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\uparrow$ $\uparrow$ free variables

Let $z=s$, $w=t$.

$$\begin{aligned} x &= \tfrac{1}{3} - 1\cdot s - \tfrac{2}{3}t \\ y &= -\tfrac{1}{3} - 1\cdot s - \tfrac{1}{3}t \\ z &= s \\ w &= t \end{aligned}$$

$$\text{General solution} = \left\{ \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} : s,t \in \mathbb{R} \right\}$$

4. $$\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 11 & 14 \\ -8 & -10 \end{pmatrix}$$

$$C \cdot A = B.$$

So

$$3\cdot(2 \quad 3) - 5(-1 \quad -1) = (11 \quad 14).$$

5. $$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \xrightarrow{-\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \xrightarrow{-2\rho_2+\rho_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Use elementary matrices.

$$\underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}_{-2\rho_2+\rho_1}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{-\rho_2}\underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}}_{-2\rho_1+\rho_2}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore,

$$2\cdot(1 \quad 2) - 1\cdot(2 \quad 3) = (0 \quad 1).$$

6.
$$\begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & 2 \\ 8 & 17 & 2 \end{pmatrix} \xrightarrow[-8\rho_1+\rho_2]{-5\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[-\rho_2+\rho_3]{-2\rho_2+\rho_1} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Use elementary matrices.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{-\rho_2+\rho_3} \underbrace{\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{-2\rho_2+\rho_1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -8 & 0 & 1 \end{pmatrix}}_{-8\rho_1+\rho_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{-5\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & 2 \\ 8 & 17 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$\Rightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ -8 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & 2 \\ 8 & 17 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 11 & -2 & 0 \\ -5 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 5 & 11 & 2 \\ 8 & 17 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So

$$-5\cdot(1\ 2\ 0) + 1\cdot(5\ 11\ 2) + 0\cdot(8\ 17\ 2) = (0\ 1\ 2)$$

7. Suppose A is $m \times n$.

Let a_{ij} be the ij-entry of A.

Fix a number k.

Let $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

be aware of the size of each matrix

$$\begin{array}{c} \\ m \end{array} \overset{n}{\begin{pmatrix} & A & \end{pmatrix}} \begin{pmatrix} {}_n C \end{pmatrix} = \begin{pmatrix} {}_m B & \end{pmatrix}$$

Then

$$\begin{aligned} c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n} &= b_1 \\ c_1 a_{21} + c_2 a_{22} + \cdots + c_n a_{2n} &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn} &= b_m \end{aligned}$$

So

$$c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

columns of A

k-th column of B.

Since k can be chosen arbitrarily,
each row of B is a linear combination of columns of A.