Sample Questions 14

1. Let $f : V \to W$ be a homomorphism. Show that f is one-to-one if and only if $\text{nullspace}(f) = \{\mathbf{0}\}$. (Equivalently, the nullity is zero.)

For the following questions, $\mathbf{A}$ is an $m \times n$ matrix and

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R}^m \\ \mathbf{x} &\mapsto \mathbf{Ax} \end{aligned}$$

is a function defined by $\mathbf{A}$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be the column vectors of $\mathbf{A}$. Note that $f(\mathbf{e}_k) = \mathbf{v}_k$ for each k.

2. Show that f is a homomorphism. Write the domain, the codomain, the range, and the null space of f in set notation. (Therefore, the range of f is the column space of A.)

3. Let
$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 1 & 1 \\ 3 & -3 & 2 & 0 \end{bmatrix}.$$
Find a basis for $\text{range}(f)$ and a basis of $\text{nullspace}(f)$. Double-check your answer by the dimension theorem:
$$\text{rank} + \text{nullity} = \dim \mathbb{R}^n = 4.$$

4. Suppose $n = 2$ and
$$f(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \text{ and } f(\mathbf{e}_2) = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}.$$
Find A.

5. Derive the formula of the rotation matrix by Problem 4. That is, find $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$ when f is the rotation by angle θ counterclockwisely; then find A.

6. Show that f is one-to-one if and only if A has full column rank. [You may use Problem 1.]

7. Show that f is onto if and only if A has full row rank.