Sample Solutions for Sample Questions 12.

1. Since $|\alpha \cup \beta| = \mathcal{B}$ and $\alpha \cap \beta = \phi$,

   we may assume $|\alpha| = r$ and $|\beta| = n-r$.

   元個數

   Since $\mathcal{B}$ is an orthogonal basis,

   $\langle \vec{v}, \vec{w} \rangle = 0$ for all $\vec{v} \in \operatorname{span} \alpha$ and $\vec{w} \in \operatorname{span} \beta$.

   Let $V = \operatorname{span} \alpha$. Then $\operatorname{span} \beta \subseteq V^{\perp}$.

   Count the dimension.

   $$\dim V = |\alpha| = r$$
   $$\dim V^{\perp} = n - \dim V = n - r$$
   $$\dim \operatorname{span} \beta = |\beta| = n - r.$$

   Since $\operatorname{span}\beta \subseteq V^{\perp}$ and $\dim V^{\perp} = \dim \operatorname{span} \beta$,

   we have $\operatorname{span}\beta = V^{\perp}$.

2. Let $\beta = \mathcal{B} \setminus \alpha$. Then $\operatorname{span}(\beta) = (\operatorname{span}(\alpha))^{\perp}$.

   Let $V = \operatorname{span}(\alpha)$

   $V^{\perp} = \operatorname{span}(\beta)$.

   Thus, $$\vec{y} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \underbrace{\sum_{\vec{v}_k \in \alpha} c_k \vec{v}_k}_{\in V} + \underbrace{\sum_{\vec{v}_k \in \beta} c_k \vec{v}_k}_{\in V^{\perp}}.$$

   So $\sum_{\vec{v}_k \in \alpha} c_k \vec{v}_k$ is the projection of $\vec{y}$ onto $\operatorname{span} \alpha$.

3.

Compute

$$\langle y, v_k \rangle = \langle c_1\vec{v}_1 + \cdots + c_n\vec{v}_n, \vec{v}_k \rangle$$

$$= c_1 \langle \vec{v}_1, \vec{v}_k \rangle + \cdots + c_n \langle \vec{v}_n, \vec{v}_k \rangle$$

都是 0，除了 $\langle \vec{v}_k, \vec{v}_k \rangle = 1$.

$$= c_k \langle \vec{v}_k, \vec{v}_k \rangle$$

$$= c_k.$$

Let $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$. Then

$$A^T \vec{y} = \begin{pmatrix} — \vec{v}_1 — \\ \vdots \\ — \vec{v}_n — \end{pmatrix} \vec{y} = \begin{pmatrix} \langle \vec{y}, \vec{v}_1 \rangle \\ \langle \vec{y}, \vec{v}_2 \rangle \\ \vdots \\ \langle \vec{y}, \vec{v}_n \rangle \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = Rep_{\mathcal{B}}(\vec{y}).$$

4. (a). Not an isomorphism

because f is not one-to-one.

e.g. $f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) Yes.

"one-to-one" and "onto".

Define $g\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} w & z-w \\ y-z & x-y \end{pmatrix}$.

Note that $g\left(f\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

and $f\left(g\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$.

So f is a bijection. (若反函數存在則為 bijection).

"preserve structure"

$$f\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}\right)$$

$$= \begin{bmatrix} a_1+a_2+b_1+b_2+c_1+c_2+d_1+d_2 \\ a_1+a_2+b_1+b_2+c_1+c_2 \\ a_1+a_2+b_1+b_2 \\ a_1+a_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1+c_1+d_1 \\ a_1+b_1+c_1 \\ a_1+b_1 \\ a_1 \end{bmatrix} + \begin{bmatrix} a_2+b_2+c_2+d_2 \\ a_2+b_2+c_2 \\ a_2+b_2 \\ a_2 \end{bmatrix}$$

$$= f\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + f\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

$$f\left(r\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} = \begin{pmatrix} ra+rb+rc+rd \\ ra+rb+rc \\ ra+rb \\ ra \end{pmatrix}$$

$$= r\cdot\begin{pmatrix} a+b+c+d \\ a+b+c \\ a+b \\ a \end{pmatrix} = r\cdot f\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(c). No, f doesn't preserve structure.

e.g. $f(1+2) = 3^3 = 27$] 不同.
$f(1)+f(2) = 1^3 + 2^3 = 9$

5. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ be a basis of V.

Let $f: V \longrightarrow \mathbb{R}^2$. That is, $f\left(a\underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\vec{v}_1} + b\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\vec{v}_2} \right) = \begin{pmatrix} a \\ b \end{pmatrix}$.
$\vec{v} \longmapsto \mathrm{Rep}_{\mathcal{B}}(\vec{v})$.

~~"one-to-one"~~
~~解方程式~~

"one-to-one"

$f(\vec{v}_1) = \begin{pmatrix} a \\ b \end{pmatrix} = f(\vec{v}_2) \Rightarrow \vec{v}_1 = \vec{v}_2 = a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

"onto"

$f\left(a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} a \\ b \end{pmatrix}$, so every vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ is in the range.

"preserve structure".

$f((a_1\vec{v}_1 + b_1\vec{v}_2) + (a_2\vec{v}_1 + b_2\vec{v}_2)) = f((a_1+a_2)\vec{v}_1 + (\cancel{a}\, b_1+b_2)\vec{v}_2)$

$= \begin{pmatrix} a_1+a_2 \\ b_1+b_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = f(a_1\vec{v}_1 + b_1\vec{v}_2) + f(a_2\vec{v}_1 + b_2\vec{v}_2)$.

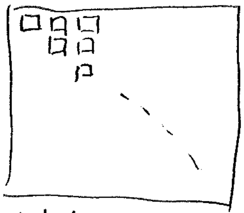
$f(r\cdot(a\vec{v}_1 + b\vec{v}_2)) = f(ra\vec{v}_1 + rb\vec{v}_2)$

$= \begin{pmatrix} ra \\ rb \end{pmatrix} = r\begin{pmatrix} a \\ b \end{pmatrix} = r\cdot f(a\vec{v}_1 + b\vec{v}_2)$.

6. Simply count the dimension.

$\dim M_{m\times n} = mn$, so $a = mn$

$\dim \cancel{S_m}\, S_n = 1+2+\cdots+n = \frac{n(n+1)}{2}$, so $b = \frac{n(n+1)}{2}$.



$\uparrow\uparrow\uparrow$

$1+2+3+\cdots+n$ 對稱矩陣能自由改變的只有這些.

7. Let $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$.

~~Then~~ Suppose $\vec{y} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

Then $\vec{y} = A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = A \cdot Rep_B(\vec{y})$.

Since A is a square matrix with $\operatorname{rank} A = n$,

A has inverse.

Take $B = A^{-1}$.

Then $Rep_B(\vec{y}) = B\vec{y}$.